Weak Dutch Books versus Strict Consistency

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Setting of the problem

$\mathcal{D} \neq \emptyset$ set of conditional gambles (conditional events), $\underline{P} : \mathcal{D} \to \mathbb{R}$ lower prevision (lower probability).

Define, $\forall n \in \mathbb{N}^+$, $\forall X_i|B_i \in \mathcal{D}$, $\forall s_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n)$,

$$\underline{G} = \sum_{i=1}^{n} s_i\, B_i\big(X_i - \underline{P}(X_i|B_i)\big) - s_0\, B_0\big(X_0 - \underline{P}(X_0|B_0)\big),$$

$$B = \bigvee_{i=0}^{n} B_i,$$

and require

$$\sup(\underline{G}|B) \geq 0. \qquad (1)$$

- $\underline{P}$ is *dF-coherent* if (1) holds, with no constraints on $s_i \in \mathbb{R}$ $(i = 0, 1, \ldots, n)$.
- $\underline{P}$ is *W-coherent* if (1) holds, provided $s_i \geq 0$ $(i = 0, 1, \ldots, n)$.
- $\underline{P}$ is *convex* if (1) holds, provided $s_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^{n} s_i = s_0 > 0$.

  A convex $\underline{P}$ is *centered convex* if, $\forall X|B \in \mathcal{D}$, it holds that $\emptyset|B \in \mathcal{D}$ and $\underline{P}(\emptyset|B) = 0$.

In each case, $\underline{G}|B$ is an *admissible gain* for $\underline{P}$.

Definition 1 (Weak Dutch Book gain). *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a convex or W-coherent lower prevision, and let $\underline{G}|B$ be an admissible gain for $\underline{P}$. $\underline{G}|B$ is a* Weak Dutch Book gain *for $\underline{P}$ if*

$$\sup(\underline{G}|B) = 0.$$

Local precision properties

Proposition 1 (The convex case). (a) *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a conditional convex lower prevision and $\underline{G}|B$ is a WDB gain, then $\exists\, P$, dF-coherent prevision, and $\alpha_P \in \mathbb{R}$ s.t., for $i = 0$ and $\forall i = 1, \ldots, n$ with $s_i > 0$, either*

$$P(B_i|B) = 0,$$

or

$$\underline{P}(X_i|B_i) = P(X_i|B_i) + \frac{\alpha_P}{P(B_i|B)}.$$

(b) *If $\underline{P}$ is unconditional, then, $\forall i = 0, 1, \ldots, n$ with $s_i > 0$,*

$$\underline{P}(X_i) = P(X_i) + \alpha_P.$$

Note that

- In the *unconditional* case, $P(B_i|B) = P(\Omega|\Omega) = 1$. The result shows that $\underline{P}$ is a 'local' translation of a precise prevision.

Proposition 2 (The W-coherent case). *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a conditional W-coherent lower prevision, $\underline{G}|B$ is a WDB gain and*

$$\mathcal{D}^+_{\underline{G}} = \{X_0|B_0\} \cup \{X_i|B_i : s_i\underline{P}(B_i|B) > 0, \text{ for } i = 1, \ldots, n\},$$

then $\underline{P}$ is a dF-coherent prevision on $\mathcal{D}^+_{\underline{G}}$.

Remark that

- The W-coherent case specialises the convex one with $\alpha_P = 0$.

Positive probability events

Proposition 3. *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a conditional W-coherent lower prevision, $\underline{G}|B$ is a WDB gain and $E|B \in \mathcal{D}$ with $\underline{P}(E|B) > 0$, then*

$$\sup(\underline{G}|B \wedge E) = 0.$$

If $\underline{P}$ is unconditional, $E = \omega \in \mathbb{P}$, domain of $\underline{G}$, then

$$\sup \underline{G} = \max \underline{G} = \underline{G}(\omega) = 0.$$

Define

$$\mathcal{P} = \{\omega \in \mathbb{P} : \underline{P}(\omega) > 0\},$$
$$\mathcal{N} = \{\omega \in \mathbb{P} : \underline{G}(\omega) = 0\}.$$

Then

$$\underline{G} \text{ WDB gain} \longrightarrow \mathcal{P} \subseteq \mathcal{N}. \qquad (2)$$

- The relationship between $\mathcal{P}$ and $\mathcal{N}$ may also depend on the *stakes* s_i, as the following example shows.

Example 1. *E_0, E_1 possible distinct events, with $E_1 \Rightarrow E_0$,*

$$\mathbb{P} = \{E_1, E_0 \wedge \neg E_1, \neg E_0\}.$$

Let $\underline{P} : \mathbb{P} \cup \{E_0\} \to \mathbb{R}$, with $\underline{P}(E_1) = \underline{P}(E_0 \wedge \neg E_1) = \underline{P}(E_0) = 0$ and $\underline{P}(\neg E_0) \in\,]0, 1]$. Then $\underline{P}$ is W-coherent and $\mathcal{P} = \{\neg E_0\}$. Let $s_i > 0$ $(i = 0, 1)$ and

$$\underline{G} = s_1\big(E_1 - \underline{P}(E_1)\big) - s_0\big(E_0 - \underline{P}(E_0)\big) = s_1\, E_1 - s_0\, E_0.$$

Then $\max \underline{G} = 0$ iff $s_1 \leq s_0$. In particular,

if $s_1 < s_0$, then $\mathcal{N} = \{\neg E_0\} = \mathcal{P}$,
if $s_1 = s_0$, then $\mathcal{N} = \{\neg E_0, E_1\} \supsetneq \mathcal{P}$.

- The converse implication of (2) does not necessarily hold (*Example 4.5* in *Corsato, Pelessoni, Vicig, 2017*).

Vulnerability to real Dutch Books

The question is: which are the agent's beliefs about suffering from real losses under *coherence* or *convexity* assumptions?

Previous results:

- If $P : \mathcal{D} \to \mathbb{R}$ is an unconditional dF-coherent probability and G is a WDB gain, then $P(G < 0) = 0$ (*Crisma, 2006*).
- If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is an unconditional W-coherent lower prevision and $\underline{G}$ is a WDB gain, then $\forall \varepsilon > 0$, $\underline{P}(\underline{G} \leq -\varepsilon) = 0$ (*Vicig, 2010*).

The general answer is the following:

Proposition 4. *If $\underline{P} : \mathcal{D} \to \mathbb{R}$ is a conditional W-coherent lower prevision and $\underline{G}|B$ is a WDB gain, then*

$$\forall \varepsilon > 0, \quad \underline{P}(\underline{G}|B \leq -\varepsilon) = 0.$$

Yet, note that, for some $\varepsilon > 0$,

- Under W-coherence, $\overline{P}(\underline{G}|B \leq -\varepsilon)$ may be > 0 (even $= 1$) (*Example 5.1* in *Corsato, Pelessoni, Vicig, 2017*);
- Under convexity, $\underline{P}(\underline{G}|B \leq -\varepsilon)$ may be > 0 (*Example 5.2* in *Corsato, Pelessoni, Vicig, 2017*).

Intermezzo

Some results follow from more general facts not involving WDBs explicitly.

Remark 1. *Let $\mathbb{P}$ be a partition of Ω, $X|B, Z|B : \mathbb{P}|B \to \mathbb{R}$ be two conditional random numbers, with $\sup(X|B) = 0$, $\sup(Z|B) < +\infty$. Suppose that $\exists\, \varepsilon > 0, \delta > 0$ s.t., $\forall \omega|B \in \mathbb{P}|B$,*

$$X|B(\omega|B) \leq -\varepsilon \quad \textit{iff} \quad Z|B(\omega|B) \geq -\delta.$$

Then $\exists\, \bar{s} > 0$ s.t, $\forall s \in\,]0, \bar{s}]$,

$$\sup(X + sZ|B) < 0.$$

Strict consistency

A possible, and actually the oldest, solution to the problem of incurring a WDB strengthens the consistency conditions, to rule out WDBs.

Definition 2 (Strict consistency). *Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be a convex (W-coherent) lower prevision. $\underline{P}$ is strictly convex (strictly W-coherent) if, for any admissible gain $\underline{G}|B$,*

$$\textit{either } \underline{G}|B = 0 \quad \textit{or } \sup(\underline{G}|B) > 0.$$

Special case: $\mathcal{A}$ algebra of unconditional events. Consider the following properties:

strict Monotonicity, (sM) :
$\forall E, F \in \mathcal{A}$, if $F \neq E \Rightarrow F$, then $\underline{P}(E) < \underline{P}(F)$;
strict Positivity, (sP) :
$\forall E \in \mathcal{A}$, if $E \neq \emptyset$, then $\underline{P}(E) > 0$;
strict Normalisation, (sN) :
$\forall E \in \mathcal{A}$, if $E \neq \Omega$, then $\underline{P}(E) < 1$.

- If $\underline{P} = P$ is dF-coherent, then
$$(sM) \longleftrightarrow (sP) \longleftrightarrow (sN);$$
- if $\underline{P}$ is W-coherent, then
$$(sM) \longleftrightarrow (sP) \longrightarrow (sN);$$
- if $\underline{P}$ is centered convex, then
$$(sM) \longrightarrow (sP) \longrightarrow (sN).$$

Proposition 5 (*Kemeny, 1955; Shimony, 1955*). *If $\mathcal{A}$ is an algebra of unconditional events and $P : \mathcal{A} \to \mathbb{R}$ is dF-coherent, then P is strictly dF-coherent iff it satisfies (sP).*

More general result:

Proposition 6. *If $\mathcal{D}$ is a set of conditional gambles s.t., $\forall$ WDB gain $\underline{G}|B \neq 0$, $\exists\, \varepsilon > 0 : (\underline{G}|B \leq -\varepsilon) \in \mathcal{D}$, non-impossible, and $\underline{P} : \mathcal{D} \to \mathbb{R}$ is W-coherent, then $\underline{P}$ is strictly W-coherent iff*

$$\forall E|B \in \mathcal{D} \setminus \{\emptyset|B\}, \quad \underline{P}(E|B) > 0.$$

Notice that

- Strict coherence is essentially confined to a *countable environment*, even with W-coherence.
- There are *alternative approaches* hedging WDBs, via desirability (*Williams, 1975; Quaeghebeur, de Cooman, Hermans, 2015*), buying/selling schemes (*Walley, 1991; Wagner, 2007*), a qualitative model (*Pedersen, 2014*).