Game solutions, probability transformations and the core

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Game theory		Imprecise probabilities
Set of players	$\Omega = \{1, \dots, n\}$	Set of values of a random variable X
(Normalized) game	$\nu : \mathcal{P}(\Omega) \to [0,1]$	Non-additive measure
Gain guaranteed to the coalition A	$\nu(A)$	Lower probability of A
Core of the game	$\mathcal{M}(\nu) := \{P \text{ additive} : P(A) \geq \nu(A)\ \forall A\}$	Credal set of ν
Shapley value	$\Phi(\nu)(i) = \sum_{T \not\ni \{i\}} (\nu(T \cup \{i\}) - \nu(T))(\lvert T\rvert!(n - \lvert T\rvert - 1)!)/n!)$	Pignistic transformation (Smets)
Banzhaf value	$B(\nu)(i) = \sum_{T \not\ni \{i\}} (\nu(T \cup \{i\}) - \nu(T))/2^{n-1}$	
Normalized Banzhaf value	$\Psi(\nu)(i) = B(\nu)(i) / \sum_{j=1}^{n} B(\nu)(j)$	

Questions:

- Is Shapley value the center of gravity (=average of the extreme points) of the core under more general conditions?
- Does it always belong to the core of the game?
- Does the (normalized) Banzhaf value make sense as a probability transformation?

ϵ-contamination models

$$\nu(A) = (1-\epsilon)P_0(A) + \epsilon \underline{P}_{\mathcal{X}}(A)$$

- $\Phi(\nu)(i) = (1-\varepsilon)P_0(\{i\}) + \frac{\varepsilon}{n}$.
- $\Psi(\nu)(i) = [(1-\varepsilon)P_0(\{i\}) + \frac{\epsilon}{2^{n-1}}]/[(1-\varepsilon) + \frac{n\varepsilon}{2^{n-1}}]$.
- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.

Minitive measures

$$\nu(A \cap B) = \min\{\nu(A), \nu(B)\}\ \forall A, B$$

- $\Phi(\nu)(i) = \sum_{j=i}^{n} \frac{m(\{1,\dots,j\})}{j}$ (Dubois, Prade).
- $\Psi(\nu)(i) = [\sum_{j=i}^{n} 2^{n-j} m(\{1,\dots,j\})]/[\sum_{j=1}^{n} j \cdot 2^{n-j} m(\{1,\dots,j\})]$.
- $\Psi(\nu) \in \mathcal{M}(\nu)$.

Pari-mutuel models

$$\nu(A) = \max\{(1+\delta)P_0(A) - \delta, 0\}$$

For $\delta < \min_i P_0(i)/(1-P_0(i))$:

- $\Phi(\nu)(i) = (1+\delta)P_0(\{i\}) - \frac{\delta}{n}$.
- $\Psi(\nu)(i) = [(1+\delta)P_0(\{i\}) - \frac{\delta}{2^{n-1}}]/[(1+\delta) - \frac{n\delta}{2^{n-1}}]$.
- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.

Belief functions

$$\nu(A) = \sum_{B \subseteq A} m(B)\ \forall A; m \geq 0, \sum_B m(B) = 1$$

- $\Psi(\nu)$ may not belong to $\mathcal{M}(\nu)$
- $\sum_i B(\nu)(i) \leq 1$
- $\Phi(\nu) = \Psi(\nu)$ on unanimity games

Comparative models on singletons

$$\mathcal{M}(\nu) = \{P : P(i) \geq P(j)\ \forall (i,j) \in \mathcal{I}\}, \nu = \min \mathcal{M}(\nu)$$

- $\Phi(\nu), \Psi(\nu) \in \mathcal{M}(\nu)$.
- $\Phi(\nu)$ may not be the center of gravity of $\mathcal{M}(\nu)$.

2-monotone capacities

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B) \forall A, B$$

- $\Phi(\nu)$ is the center of gravity of the core (Shapley).
- $\Psi(\nu)$ may not belong to $\mathcal{M}(\nu)$ if $\lvert\Omega\rvert \geq 4$.
- $\Psi(\nu) \in \mathcal{M}(\nu)$ if $\lvert\Omega\rvert = 3$.

Coherent lower probabilities

$$\nu(A) = \min\{P(A) : P \in \mathcal{M}(\nu)\}\ \forall A$$

- $\Phi(\nu) \in \mathcal{M}(\nu)$ if $\lvert\Omega\rvert \leq 4$.
- $\Phi(\nu)$ may not belong to $\mathcal{M}(\nu)$ if $\lvert\Omega\rvert \geq 5$ (Baroni, Vicig).

Lower probabilities avoiding sure loss

$$\mathcal{M}(\nu) \neq \emptyset$$

- $\Phi(\nu), \Psi(\nu)$ may not belong to $\mathcal{M}(\nu)$.

Essential references

- P. Baroni, P. Vicig, *An uncertainty interchange format with imprecise probabilities.* Int. J. of Approximate Reasoning, 2005.
- D. Dubois, H. Prade, *Unfair coins and necessity measures: towards a possibilistic interpretation of histograms.* Fuzzy Sets and Systems, 1983.
- P. Smets, *Decision making in the TBM: the necessity of the pignistic transformation.* Int. J. of Approximate Reasoning, 2005.
- L. Shapley, *Cores of convex games.* Int. J. of Game Theory, 1971.

At a glance

- Problem: studying game solutions as probability transformations.
- Idea: use the normalized Banzhaf value and Shapley value, and check if the latter is the center of gravity of the core in other conditions than for 2-monotone games.
- Results: they belong to the core in a number of cases, and can be given simpler expressions when ν satisfies additional conditions.