A study of the Pari-Mutuel Model from the point of view of Imprecise Probabilities

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Introduction

The Pari-Mutuel model...

-...is a distortion model...
-...originated in horse racing...
-...applied to finance and risk analysis.

A PMM is defined by...

-P_0: a precise probability.
-$\delta > 0$ a taxation from the house.

A PMM (P_0, δ) defines a coherent lower and upper probability by...

$$\underline{P}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\},$$
$$\overline{P}(A) = \min\{(1+\delta)P_0(A), 1\}.$$

Some properties and notations...

-$\underline{P}$ is 2-monotone and $\overline{P}$ is 2-alternating.
-$\overline{P}(A) - \underline{P}(A) \leq \delta$.
-When $\overline{P}(A) < 1$, $\overline{P}$ is additive:

$$\overline{P}(A) = \sum_{x \in A} \overline{P}(\{x\}).$$

-$\mathcal{M}(P_0, \delta)$ denotes the credal set of the lower and upper probability $\underline{P}, \overline{P}$ defined from a PMM (P_0, δ).

We aim to...

-...investigate the PMM within the framework of Imprecise Probabilities.

In particular...

1. What are the connections between the PMM and other models in IP Theory?
2. What can we say about the extreme points of $\mathcal{M}(P_0, \delta)$?
3. How can we merge information given in terms of PMMs?

First contribution: Connection with other IP models

A *probability interval* is an IP model that is determined by the restriction to singletons:

$$\mathcal{I} = \{[l_i, u_i] : i = 1, \ldots, n\}.$$

Its associated credal set is $\mathcal{M}(\mathcal{I}) = \{P \text{ prob.} \mid l_i \leq P(\{x_i\}) \leq u_i \ \forall i = 1, \ldots, n\}$.
A PMM is a probability interval: Let $\underline{P}, \overline{P}$ be the PMM induced by P_0, δ. Define the probability interval $\mathcal{I} := \{[\underline{P}(\{x_i\}), \overline{P}\{x_i\}] \mid i = 1, \ldots, n\}$. Then $\mathcal{M}(\mathcal{I}) = \mathcal{M}(P_0, \delta)$.
Connection with belief functions: Let $\underline{P}$ be the lower probability induced by a PMM (P_0, δ), and let $k = \min\{|A| : \underline{P}(A) > 0\}$. $\underline{P}$ is a belief function if and only if either:

(B1) $k = n$.

(B2) $k = n-1$ and $\sum_{i=1}^{n} \underline{P}(\mathcal{X} \setminus \{x_i\}) \leq 1$.

(B3) $k < n-1$, $\exists! B$ with $|B| = k$ and $\underline{P}(B) > 0$, and $\underline{P}(A) > 0 \iff B \subseteq A$.

(B4) $k < n-1$, $\exists! B$ with $|B| = k-1$ and $\delta = \frac{P_0(B)}{1-P_0(B)}$, and $\underline{P}(A) > 0 \iff B \subset A$.

Second contribution: Extreme points of $\mathcal{M}(P_0, \delta)$

Form of the extreme points: Let $\underline{P}, \overline{P}$ be the PMM induced by P_0, δ, and let S^n be the set of permutations of $\{1, \ldots, n\}$. Then $ext(\mathcal{M}(P_0, \delta)) = \{P_\sigma : \sigma \in S^n\}$, where:

$$P_\sigma(\{x_{\sigma(i)}\}) = \begin{cases} \overline{P}(\{x_{\sigma(i)}\}) & \text{for } i < j \\ \underline{P}(\{x_{\sigma(j)}, \ldots, x_{\sigma(n)}\}) & \text{for } i = j \\ 0 & \text{for } i > j, \end{cases}$$

and where j satisfies $\overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(j-1)}\}) < \overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(j)}\}) = 1$.
Maximal number of extreme points: The maximum number of extreme points of $\mathcal{M}(P_0, \delta)$ is

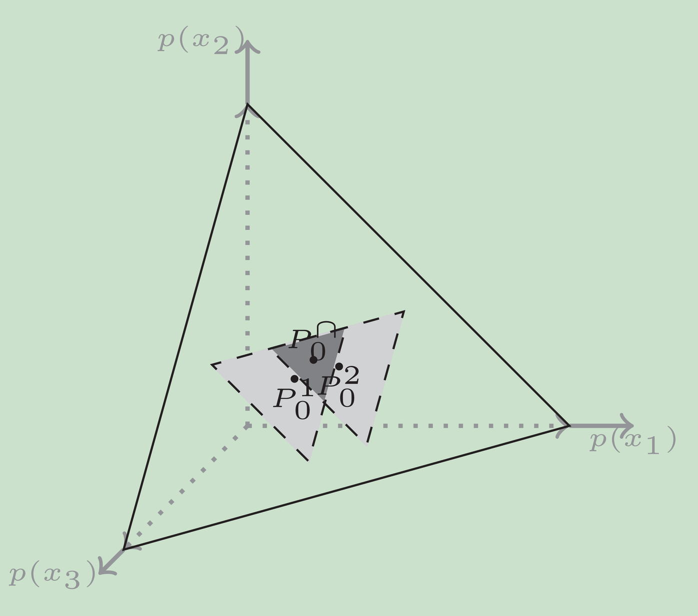
$$\frac{n}{2}\binom{n}{\frac{n}{2}} \text{ if } n \text{ is even} \qquad \text{and} \qquad \frac{n+1}{2}\binom{n}{\frac{n+1}{2}} \text{ if } n \text{ is odd.}$$

↪ It coincides with the maximum number of extreme points for probability intervals!
Moreover, if we denote $\mathcal{L} = \{A \subseteq \mathcal{X} \mid \overline{P}(A) = 1\}$, it holds that

$$|ext(\mathcal{M}(P_0, \delta))| \leq \sum_{A \in \mathcal{L}} \left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right|.$$

Third contribution: Information fusion

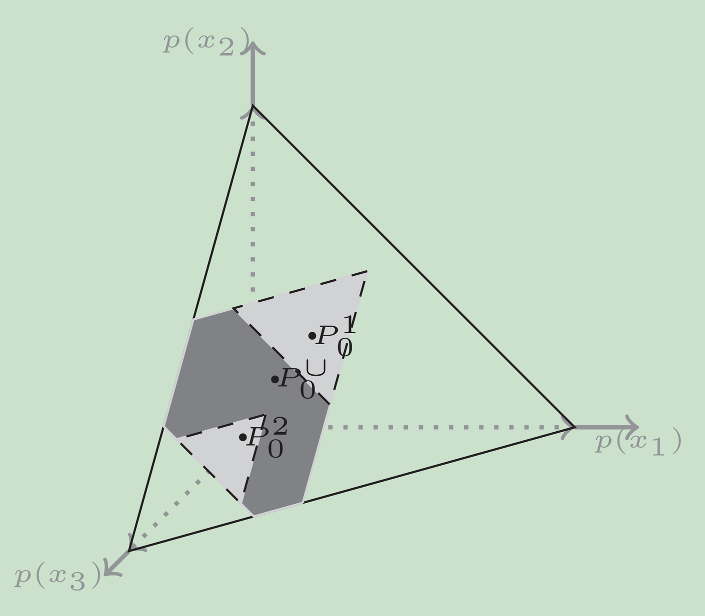
Case 1: conjunction



$\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)$

If $\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2) \neq \emptyset$, it is associated with a PMM.

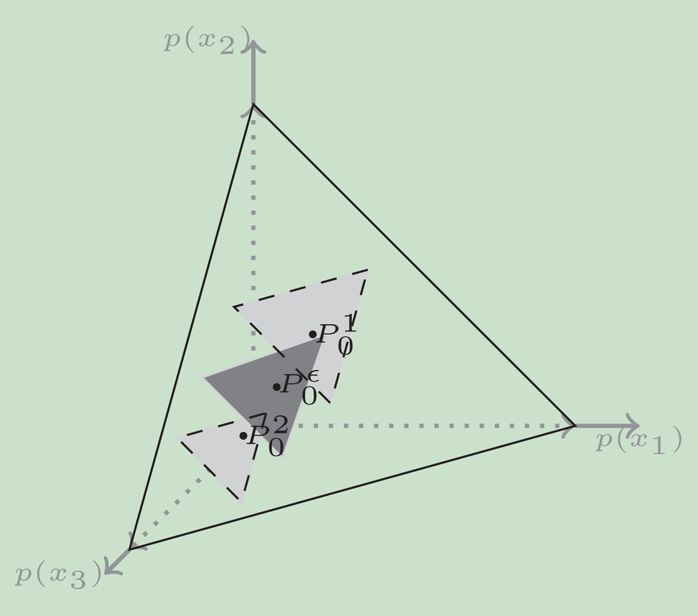
Case 2: disjunction



$\mathcal{M}(P_0^{\cup}, \delta^{\cup})$

Neither $\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)$ nor $conv\left(\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)\right)$ are induced by a PMM. However, the latter can be outer-approximated by a PMM.

Case 3: mixture



$\epsilon\mathcal{M}(P_0^1, \delta_1) + (1-\epsilon)\mathcal{M}(P_0^2, \delta_2)$

The convex combination of two PMMs gives rise to another PMM.

Conclusions and references

At a glance

- PMM: a probability interval with bounded imprecision $\overline{P}(A) - \underline{P}(A) \leq \delta$.
- Maximal number of extreme points: the same than for probability intervals.
- Extreme points: easy to compute and bounded in number.

References

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