Options We choose between abstract options, collected in a vector space $\mathcal{F}$, ordered by a reflexive vector ordering $\preceq$, whose irreflexive part is defined by $u \succ u \iff (u < v \wedge u \neq v)$ for all $u$ and $v$ in $\mathcal{F}$. $\mathcal{F}$ is the collection of non-empty but finite subsets of $\mathcal{U}$.

Choice functions A choice function on $\mathcal{F}$ is a map

$C : \mathcal{F} \to \mathcal{P}(\mathcal{F}) \setminus \{\emptyset\}$

such that $C(A) \subseteq A$.

Rationality axioms We call a choice function $C$ on $\mathcal{F}$ coherent if for all $A, B, C \in \mathcal{F}$, $A \subseteq B$, $A \subseteq C$, and $B \cap C = \emptyset$, then $C(A) \subseteq (C(B) \cap C(C))$.

A useful map Use the special coherent set of indifferent options $I$, to find alternative expressions for the equivalent classes $[u] = \{u + v : v \in C(A)\}$, and the vector ordering $\leq$ on $\mathcal{F}^2$.

A useful map $\text{CoM}: \mathcal{F}^2 \to \mathcal{F}$.

To define coherent choice functions, we only need an ordered linear space.

Is there a de Finetti-like Representation theorem?

Polynomial gambles Consider the $3^*$-space $\mathcal{S}_N = \{0 \in \mathbb{R}^3 : \sum_0^N 1 = 1\}$, and the linear space $\mathcal{F}^*(\mathcal{S}_N)$ of polynomial gambles $x$ on $\mathcal{S}_N$ that are restricted to $\mathcal{S}_N$ of multivariate polynomials $p$ on $\mathbb{R}^3$, in the sense that $n(0) = p(0)$ for all $n \in \mathcal{F}^*$.

Bernstein gambles For any $n \in \mathbb{N}$ and any $m \in \mathbb{R}$, the Bernstein basis polynomial $B_n^m$ on $\mathcal{S}_N$ is given by $B_n^m(|t|) := \sum_{i=0}^{n} \binom{n}{i} m^i (1-m)^{n-i}$ for all $n \in \mathbb{R}^3$. The restriction to $\mathcal{S}_N$ is called a Bernstein gamble, which we also denote by $B_n^m : \mathcal{S}_N \to \mathcal{F}$.

Is there a representation that does not depend on counts?

Cylindrical extension In case of infinitely many exchangeable $X_i$, the global possibility space is $\mathcal{N}^\infty$. We identify any gamble $f$ on $\mathcal{S}_N$ with its cylindrical extension

$\text{Cyl}(f) : \{\mathcal{S}_i \in \mathcal{N}^\infty : i \in \mathbb{N}\} \to \mathcal{S}_N$.

for all $i \in \mathbb{N}$. Using this convention, we can identify $\text{Cyl}(f)$ with a subset in $\mathcal{S}_N$. Gambles of finite structure We will call any gamble that depends only on a finite number of variables a gamble of finite structure. We collect all such gambles in $\mathcal{S}_N^{(\infty)} = \{f \in \mathcal{S}_N^{(\infty)} : \text{all } f_i \in \mathcal{S}_N^{(\infty)}\}$. The subject assesses the sequence of variables $X_1, X_2, \ldots$ to be exchangeable; he is indifferent between any gamble $f$ on $\mathcal{S}_N$ and its permuted variant $f^\pi$, for any $\pi$ in $\mathcal{S}_N^{\infty}$; and the sequence $X_1, X_2, \ldots$ is exchangeable, in the sense that it is an infinite sequence of independent and identically distributed random variables.

What about the countable case?

Theorem 4 (Countable Representation). A choice function $C$ on $\mathcal{S}_N^{(\infty)}$ is countable (countably exchangeable if $C$ is compatible with $\mathcal{S}_N^{(\infty)}$ if and only if there is a unique representing choice function $C$ on $\mathcal{S}_N^{(\infty)}$ such that

$C(A) = \{f : A \cap C(A) - C(A)\}$

for all $A \in \mathcal{S}_N^{(\infty)}$. Furthermore, in that case, $C$ is given by $C(A) = H(C(A)) - H(C(A))$ for all $A \in \mathcal{S}_N^{(\infty)}$. Finally, $C$ is coherent if and only if $C$ is.

Can we add assessments?

Category permutation invariance Suppose that, in addition to exchangeability, the subject also has reason not to distinguish between the different elements of $\mathcal{F} := \{1, \ldots, k\}$: consider any permutation of $\mathcal{F}$ and any outcome $X_i \in \{X_1, \ldots, X_k\}$, then he has reason not to distinguish between $X_i$ and $X_{i'}$ for any $i, i' \in \mathcal{F}$. With any gamble $f$ on $\mathcal{F}$ there corresponds a permuted gamble $\tilde{f}$, given by $\tilde{f}(i) = f(i')$ for all $i \in \mathcal{F}$.