A Two-Tiered Propositional Framework for Handling Multisource Inconsistent Information

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Introduction

- Belnap 4-valued logic is the oldest approach to reasoning under *incomplete and inconsistent information due to several sources*
- It combines two 3-valued logics:
  - Kleene logic (of incomplete information)
  - Priest logic of Paradox (inconsistent information)
- We have already shown that the two later logics can be captured by MEL, a fragment of the KD modal logic, with semantics in terms of subsets of interpretations (JANCL, 2013)
- This translation makes formulas easier to understand and shows the limitation of expressiveness of these truth-functional logics.

It is natural to do the same with Belnap logic

Outline

Belnap set up for multisource information

- A computer receives information about atomic propositions from outside sources
- Each source asserts whether each atomic proposition is true, false, or is silent
- An epistemic set-up assigns one of four values $\mathbb{V}_4 = \{\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathbf{U}\}$ to each atomic proposition
  - **T**: computer has been told that *a* is true (1) by at least one source, and false (0) by none
  - **F**: computer has been told that *a* is false by at least one source, and true by none
  - **C**: computer has been told that *a* is true by one source and false by another
  - **U**: computer has been told nothing about *a*

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Connectives of Belnap 4-valued logic

One of four values can be truth-functionally assigned to all propositions via negation, conjunction and disjunction:

$\vee$	**F**	**U**	**C**	**T**
F	**F**	**U**	**C**	**T**
U	**U**	**U**	**T**	**T**
C	**C**	**T**	**C**	**T**
T	**T**	**T**	**T**	**T**

$\wedge$	**F**	**U**	**C**	**T**
F	**F**	**F**	**F**	**F**
U	**F**	**U**	**F**	**U**
C	**F**	**F**	**C**	**C**
T	**F**	**U**	**C**	**T**

a	$\neg a$
F	**T**
U	**U**
C	**C**
T	**F**

- The information ordering, $\sqsubset$ whose meaning is “less informative than”, such that **U** $\sqsubset$ **T** $\sqsubset$ **C**; **U** $\sqsubset$ **F** $\sqsubset$ **C**
- The truth ordering, $<_t$, representing “more true than” according to which **F** $<_t$ **C** $<_t$ **T** and **F** $<_t$ **U** $<_t$ **T**

Syntax

- A standard propositional language $\mathcal{L}$ with variables $V = \{a, b, c, \dots\}$ and connectives $\wedge, \vee, \neg$
- Belnap 4-valued logic has no tautologies, hence no axioms
- It has a proof system defined by a set of 15 inference rules describing the properties of a De Morgan algebra
- Formulas can be put in normal form as a conjunction of clauses, i.e., $p = p_1 \wedge \dots \wedge p_n$, where the p_i's are disjunctions of literals

Semantics

- A Belnap valuation is a mapping vb from formulas $\mathcal{L}$ to the 4 values in $\mathbb{V}_4$, equipped with a bilattice structure
- Let $\Gamma \subseteq \mathcal{L}$ and $p \in \mathcal{L}$, then we define the consequence relation by means of the truth ordering $\leq_t$ as

$$\Gamma \vDash_B p \quad \text{iff} \quad \exists p_1, \ldots, p_n \in \Gamma,\ \forall vb,\ vb(p_1) \wedge \ldots \wedge vb(p_n) \leq_t vb(p)$$

- Designated truth-values **C, T**:

$$\Gamma \vDash_B p \quad \text{iff} \quad (\forall i, vb(p_i) \in \{\mathbf{C}, \mathbf{T}\}) \Rightarrow vb(p) \in \{\mathbf{C}, \mathbf{T}\}$$

Belnap logic is sound and complete with respect to Belnap semantics: $\Gamma \vdash_B p$ iff $\Gamma \vDash_B p$ using the 15 rules

Three-valued fragments

- Kleene logic of incomplete information: obtained from Belnap logic by deleting the truth-value **C** and keeping designated truth-value **T**
- Priest logic of conflicting information: obtained from Belnap logic by deleting the truth-value **U** , keeping **C, T** as designated
- they have the same truth-tables

Syntax

- Kleene logic has one more inference rule: $q \wedge \neg q \vdash p \vee \neg p$
- Priest logic is Belnap logic plus axiom $p \vee \neg p$

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Generalised Multi-source set-up

- n sources provide information in the form of epistemic states $E_i \subseteq \Omega$, where $\Omega =$ set of interpretations of the language

  For source i the real state of affairs w should lie in E_i

- A Boolean capacity β can be built from these pieces of information $\{E_1, E_2, \ldots, E_n\}$ as :

  $$\beta(A) = 1 \text{ if } \exists i E_i \subseteq A, \text{ 0 otherwise.}$$

- β is a monotonic set-function and the least sets F with $\beta(F) = 1$ are called focals of β.
- If $A = [p]$, $\beta(A) = 1$ means that there is at least one source i that believes that p is true

Generalized Belnap epistemic values

There are four epistemic statuses extending Belnap's **T, F, U, C** assigned to all propositions p:

- Support of p: $\mathbf{T} \equiv (\beta([p]) = 1 \text{ and } \beta([\neg p]) = 0)$.
  p is asserted by at least one source and negated by no other one.
- Rejection of p: $\mathbf{F} \equiv (\beta([\neg p]) = 1 \text{ and } \beta([p]) = 0)$.
  p is negated by at least one source and asserted by no other one.
- Ignorance about p: $\mathbf{U} \equiv (\beta([p]) = \beta([\neg p]) = 0)$
  No source supports nor negates p.
- Conflict about p: $\mathbf{C} \equiv (\beta([p]) = \beta([\neg p]) = 1)$
  Some sources assert p, some negate it.

Difference with Belnap: it is not truth-functional

Important special cases

- β is minitive, i.e., $\beta(A \cap B) = \min(\beta(A), \beta(B))$
  It is then a necessity measure and $\mathcal{F}_\beta = \{E\}$.
  There is only one source and its
  information is incomplete, but there is no conflict
- the focal sets are singletons $\{e_i\}$. Then β is maxitive, i.e., $\beta(A \cup B) = \max(\beta(A), \beta(B))$
  All sources have complete information, so there are
  conflicts, but no ignorance

Syntax of BC

- A higher level propositional language $\mathcal{L}_\Box$ on top of $\mathcal{L}$, whose formulas are denoted by Greek letters $\phi, \psi, \ldots$
- Defined by:
  - if $p \in \mathcal{L}$ then $\Box p \in \mathcal{L}_\Box$
  - if $\phi, \psi \in \mathcal{L}_\Box$ then $\neg\phi \in \mathcal{L}_\Box$, $\phi \wedge \psi \in \mathcal{L}_\Box$
- Note that the language $\mathcal{L}$ is not part of $\mathcal{L}_\Box$
  since atomic variables of $\mathcal{L}_\Box$ are of the form $\Box p, p \in \mathcal{L}$
- As usual $\Diamond p$ stands for $\neg\Box\neg p$

Syntax: axioms

It is a two-tiered propositional logic plus some modal axioms

1. All axioms of propositional logics for $\mathcal{L}_\Box$-formulas.
2. The modal axioms:
   - *(RM)* $\Box p \to \Box q$ if $\vdash p \to q$ in propositional logic.
   - *(N)* $\Box p$, whenever p is a propositional tautology.
   - *(P)* $\Diamond p$, whenever p is a propositional tautology.

The only rule is modus ponens: If ψ and $\psi \to \phi$ then ϕ

The two dual modalities $\Box$ and $\Diamond$ play the same role

Semantics

A BC-model of an atomic formula $\Box p$ is a B-capacity β
The satisfaction of BC-formulas is defined as:

- $\beta \models \Box p$, if and only if $\beta([p]) = 1$
- $\beta \models \neg\phi$, $\beta \models \phi \wedge \psi$ in the standard way

BC logic is sound and complete wrt B-capacity models

Outline

Translation of elementary propositions

Principle

$\Box a$ is equated to $vb(a) \geq_t \mathbf{C}$, i.e., at least one source supports a
$\Box \neg a$ is equated to $vb(a) \leq_t \mathbf{C}$, i.e., at least one source supports $\neg a$

Denoting by $\mathcal{T}(vb(a) \in \Theta \subseteq \mathbb{V}_4)$ the translation of a partial Belnap value assignment:

$$\mathcal{T}(vb(a) = \mathbf{T}) = \Box a \wedge \Diamond a \qquad \mathcal{T}(vb(a) = \mathbf{F}) = \Box \neg a \wedge \Diamond \neg a$$
$$\mathcal{T}(vb(a) = \mathbf{U}) = \Diamond a \wedge \Diamond \neg a \qquad \mathcal{T}(vb(a) = \mathbf{C}) = \Box a \wedge \Box \neg a$$

Translation of complex propositions

We use Belnap truth-tables

- Negation
  $\mathcal{T}(vb(\neg p) = \mathbf{T}) = \mathcal{T}(vb(p) = \mathbf{F})$
  For $\mathbf{x} \in \{\mathbf{U}, \mathbf{C}\}$,
  - $\mathcal{T}(vb(\neg p) \geq_t \mathbf{x}) = \mathcal{T}(vb(p) \leq_t \mathbf{x})$
  - $\mathcal{T}(vb(\neg p) = \mathbf{x}) = \mathcal{T}(vb(p) = \mathbf{x})$
- Conjunction
  $\mathcal{T}(vb(p \wedge q) = \mathbf{T}) = \mathcal{T}(vb(p) = \mathbf{T}) \wedge \mathcal{T}(vb(q) = \mathbf{T})$
- Disjunction
  $\mathcal{T}(vb(p \vee q) = \mathbf{T}) =$
  $\mathcal{T}(vb(p) = \mathbf{T}) \vee \mathcal{T}(vb(q) = \mathbf{T})$
  $\vee(\mathcal{T}(vb(p) = \mathbf{U}) \wedge \mathcal{T}(vb(p) = \mathbf{C})) \vee (\mathcal{T}(vb(p) = \mathbf{C}) \wedge \mathcal{T}(vb(p) = \mathbf{U}))$

Results

This translation of Belnap logic reaches the following fragment of BC-language:

$$\mathcal{L}^B_\Box = \Box a | \Box \neg a | \phi \wedge \psi | \phi \vee \psi$$

without negation in front of $\Box$ and only literals inside $\Box$

Theorem

Let $\phi \vdash_{Belnap} \psi$ be any of the 15 inference rules of Belnap logic

Then, the following inference rule is valid in BC:

$$\mathcal{T}(vb(\psi) \geq_t \mathbf{C}) \vdash_{BC} \mathcal{T}(vb(\phi) \geq_t \mathbf{C})$$

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Belnap valuations and multisource epistemic states

- A Belnap valuation corresponds to a set of information items $\{E_1, \ldots, E_n\}$, where
  - $E_i = [(\bigwedge_{a \in T_i} a) \wedge (\bigwedge_{b \in F_i} \neg b)]$ is a partial model
  - $T_i =$ atomic propositions declared true by source i
  - $F_i =$ atomic propositions declared false by source i
- $vb \longrightarrow (T_i, F_i)_{i=1}^n \longrightarrow$ atomic Boolean capacity α_{vb} with focals

$$\{[a] : a \in \cup_{i=1}^n T_i\} \cup \{[\neg b] : b \in \cup_{i=1}^n F_i\}$$

  defined by literals.
- Conversely a set of information items $\{E_1, \ldots, E_n\}$ can be mapped to a Belnap valuation vb_β, with $T_i = \{a : E_i \subseteq [a]\}$ and $F_i = \{b : E_i \subseteq [\neg b]\}$

Results

Definition: Two capacities are Belnap-equivalent if they map to the same Belnap valuation $vb_\beta = vb_{\beta'}$

Proposition

For any B-capacity β, there exists an atomic B-capacity α Belnap-equivalent to it.

Our translation of Belnap logic into BC is consequence-preserving !

Main Theorem

Let Γ be a set (conjunction) of formulas in propositional logic interpreted in Belnap logic, and p be another such formula. Then

$$\Gamma \vdash_B p \quad \text{if and only if} \quad \{\mathcal{T}(vb(q) \geq_t \mathbf{C}) : q \in \Gamma\} \vdash_{BC} \mathcal{T}(vb(q) \geq_t \mathbf{C})$$

Conclusion

- Our translation recovers the previous translations of Kleene logic (by adding rule $\Box q \wedge \Box\neg q \vdash \Box p \vee \Box\neg p$) and Priest logic (adding axiom $\Box p \vee \Box\neg p$)
- Boolean capacities can synthetize general Boolean inconsistent information: extend to valued capacities ?
- The logic of capacities may provide a general framework for inconsistency management
  - paraconsistent logics
  - Avron-Ben-Naim approach to multi-source information handling
  - Minimal consistent subsets : $\beta(A \cap B) = \min(\beta(A), \beta(B))$ if $A \cap B \neq \emptyset$
  - Argument ranking methods

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