

**Efficient computation  
of updated lower expectations for  
imprecise continuous-time hidden  
Markov chains**





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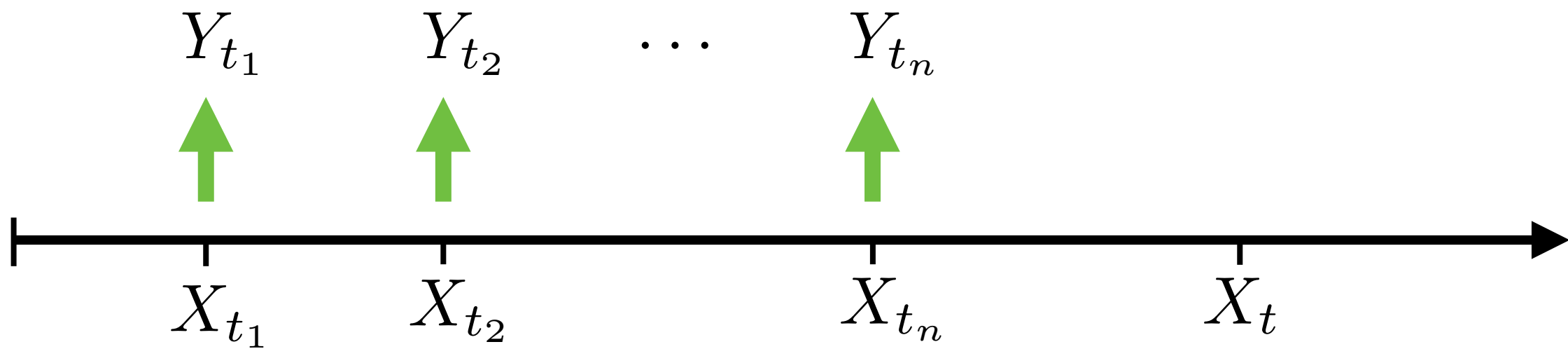


# Imprecise continuous-time Markov chain



# Imprecise continuous-time Markov chain

hidden

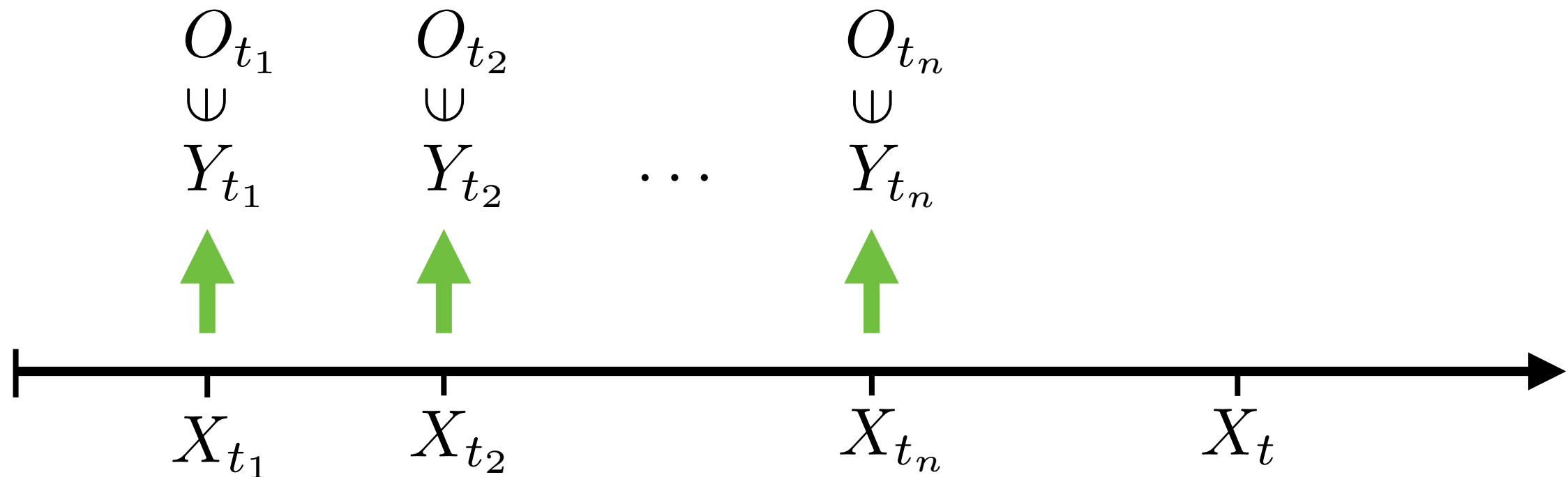


# Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} \in O_{t_1}, \dots, Y_{t_n} \in O_{t_n})$$



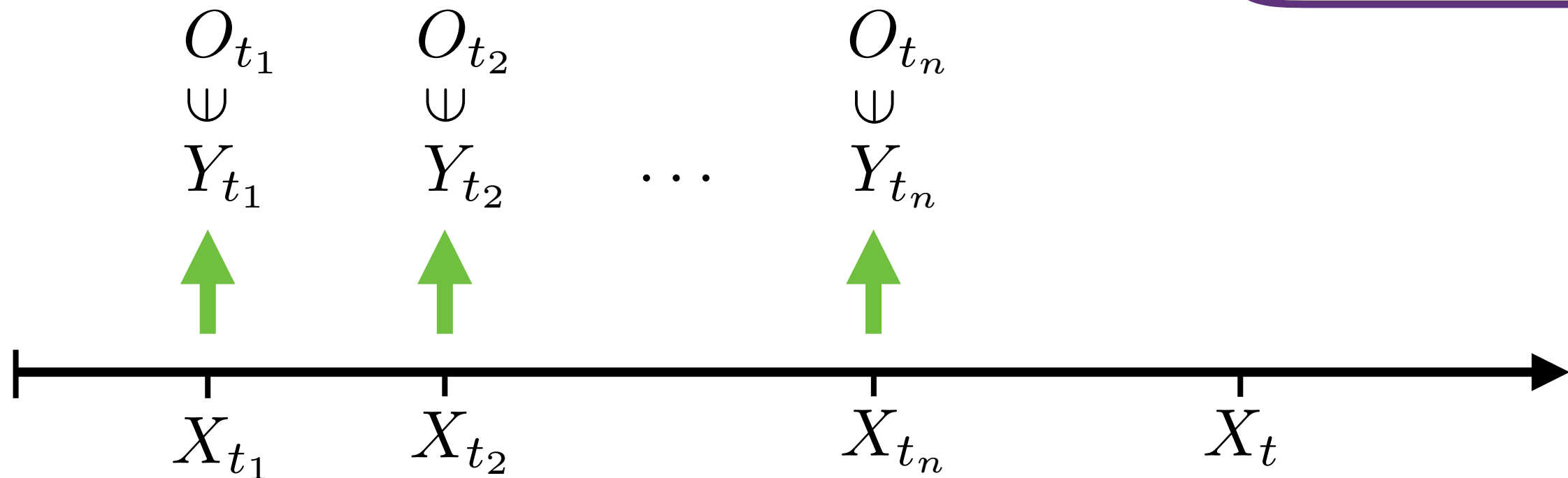
# Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} \in O_{t_1}, \dots, Y_{t_n} \in O_{t_n})$$

efficient algorithms



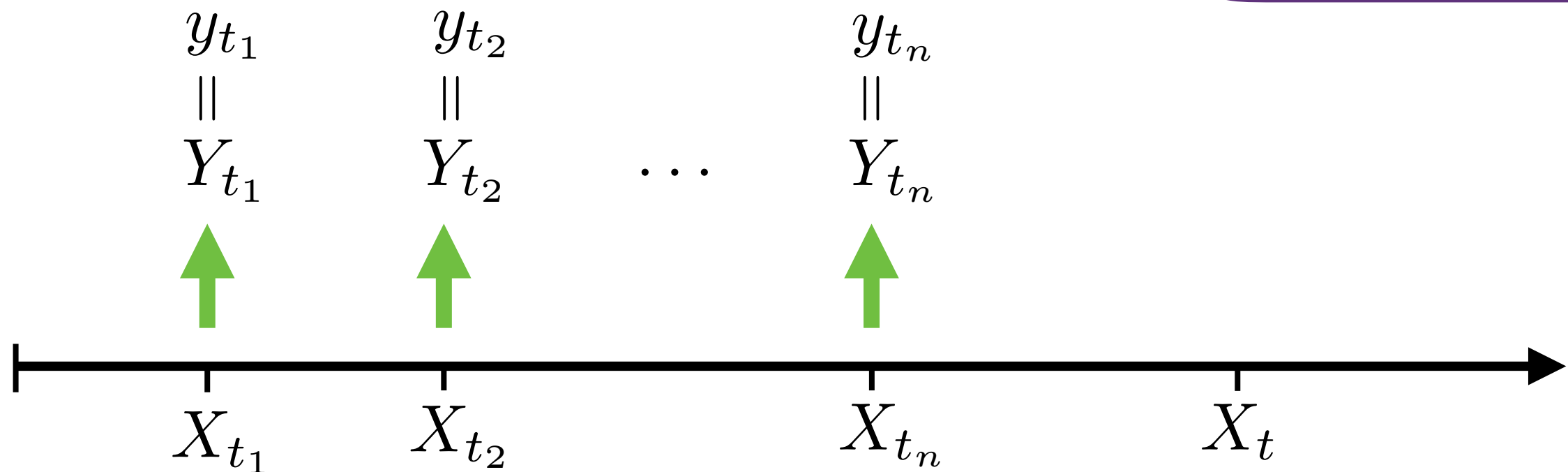
# Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} = y_{t_1}, \dots, Y_{t_n} = y_{t_n})$$

point observations



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more?



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**Efficient Computation of Updated Lower Expectations for Imprecise Continuous-Time Hidden Markov Chains**  
Thomas Krak, Jasper De Bock, Arno Siebes

**Abstract** We consider the problem of performing inference with *imprecise continuous-time hidden Markov chains*, that is, *imprecise continuous-time Markov chains* that are augmented with random *output variables* whose distribution depends on the hidden state of the chain. The prefix 'imprecise' refers to the fact that we do not consider a classical continuous-time Markov chain, but replace it with a robust extension that allows us to represent various types of model uncertainty, using the theory of *imprecise probabilities*. The inference problem amounts to computing lower expectations of functions on the state-space of the chain, given observations of the output variables. We develop and investigate this problem with very few assumptions on the output variables; in particular, they can be chosen to be either discrete or continuous random variables. Our main result is a polynomial runtime algorithm to compute the lower expectation of functions on the state-space at any given time-point, given a collection of observations of the output variables.

**"Precise" Continuous-Time Markov Chains**  
State-space  $X$  (e.g.,  $X = \{\text{healthy, sick}\}$ )  
Continuous-time Markov chain  $P$  specifies r.v.  $X_t$  at each time  $t \in \mathbb{R}_{\geq 0}$   
For any finite number of time-points, e.g.  $0 < t < r < s$ ,  $P$  induces a Bayesian network:  
 $X_0 \rightarrow X_t \rightarrow X_r \rightarrow X_s$   
Satisfies Markov property:  $P(X_t | X_0, X_r, X_s) = P(X_t | X_r)$

**Imprecise Continuous-Time Markov Chains**  
Now a set  $\mathcal{P}$  of distributions.  
Each  $P \in \mathcal{P}$  specifies r.v.  $X_t$  at each time  $t \in \mathbb{R}_{\geq 0}$   
For any finite number of time-points, e.g.  $0 < t < r < s$ ,  $\mathcal{P}$  induces a credal network:  
 $X_0 \rightarrow X_t \rightarrow X_r \rightarrow X_s$   
Satisfies imprecise Markov property:  $\underline{P}(X_t | X_0, X_r, X_s) = \underline{P}(X_t | X_r)$

**Imprecise CT Hidden Markov Chains**  
States  $X_t$  cannot be directly observed. Instead we observe  $Y_t$ , which "correlates" with  $X_t$  (e.g., symptoms of a disease).  
 $X_0 \rightarrow X_t \rightarrow X_r \rightarrow X_s$   
 $Y_0 \rightarrow Y_t \rightarrow Y_r \rightarrow Y_s$   
For simplicity, we use a *precise, homogeneous output model*:  
 $\underline{P}(Y_t | X_t) = P(Y_t | X_t) = P(Y | X), t \in \mathbb{R}_{\geq 0}$   
We are interested in inferences about the states given observations.  
For example, given some  $O \subseteq Y$ , we want to know  $\underline{E}[f(X_t) | Y_t \in O]$ .

**Outputs with Positive (Upper) Probability**  
If the observation  $(Y_t \in O)$  has positive probability, we use Bayes' rule:  
 $E_p[f(X_t) | Y_t \in O] = \sum_{x \in X} f(x) \frac{P(X_t = x, Y_t \in O)}{P(Y_t \in O)}$   
For the imprecise model, we use *regular extension*:  
 $\underline{E}[f(X_t) | Y_t \in O] = \inf\{E_p[f(X_t) | Y_t \in O] : P \in \mathcal{P}, P(Y_t \in O) > 0\}$ ,  
whenever  $\bar{P}(Y_t \in O) > 0$ .  
This lower expectation satisfies a *generalised Bayes' rule*:  
 $\underline{E}[f(X_t) | Y_t \in O] = \max\{\mu \in \mathbb{R} : \underline{E}[P(Y_t \in O | X_t)(f(X_t) - \mu)] \geq 0\}$

**Continuous Outputs**  
If  $Y_t$  is continuous, then usually  $P(Y_t = y) = 0$  for all  $P \in \mathcal{P}$ . Assume a (conditional) probability density function  $\phi: Y \times X \rightarrow \mathbb{R}$ :  
 $P(Y_t \in O | X_t = x) = \int_O \phi(y|x) dy$   
Take a sequence  $(O_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} O_n = O$ . Then define  
 $E_p[f(X_t) | Y_t \in y] = \lim_{n \rightarrow \infty} E_p[f(X_t) | Y_t \in O_n]$   
This limit exists under suitable assumptions; if  $E_p[\phi(y | X_t)] > 0$ :  
 $E_p[f(X_t) | Y_t = y] = \frac{E_p[f(X_t)\phi(y | X_t)]}{E_p[\phi(y | X_t)]}$

**Continuous Outputs, Imprecise Case**  
For the imprecise case, when  $\underline{E}[\phi(y | X_t)] > 0$  we define  
 $\underline{E}[f(X_t) | Y_t = y] = \inf\{E_p[f(X_t) | Y_t = y] : P \in \mathcal{P}\}$   
This lower expectation satisfies a limit interpretation  
 $\underline{E}[f(X_t) | Y_t = y] = \lim_{n \rightarrow \infty} \underline{E}[f(X_t) | Y_t \in O_n]$   
and a *generalised Bayes' rule for (finite) mixtures of densities*:  
 $\underline{E}[f(X_t) | Y_t = y] = \max\{\mu \in \mathbb{R} : \underline{E}[\phi(y | X_t)(f(X_t) - \mu)] \geq 0\}$

**Solving the Generalised Bayes' Rule(s)**  
In both cases, we have a generalised Bayes' rule:  
 $\underline{E}[f(X_t) | Y_t \in O] = \max\{\mu \in \mathbb{R} : \underline{E}[P(Y_t \in O | X_t)(f(X_t) - \mu)] \geq 0\}$   
 $\underline{E}[f(X_t) | Y_t = y] = \max\{\mu \in \mathbb{R} : \underline{E}[\phi(y | X_t)(f(X_t) - \mu)] \geq 0\}$   
See the paper for a polynomial runtime algorithm to solve these.

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