A study of the Pari-Mutuel Model from the point of view of Imprecise Probabilities

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Overview

Pari-Mutuel Model

Connection between the PMM and other models
  Connection with probability intervals
  Connection with belief functions

Number of extreme points induced by a PMM

Information fusion with PMMs

Conclusions
The Pari-Mutuel Model

Distortion model.

Originated in horse racing.

Applied in finance, risk analysis, . . .

Precise probability

\[ P_0 \]

Taxation from the house

\[ \delta > 0 \]

\[
\begin{align*}
\overline{P}(A) &= \max\{(1 + \delta)P_0(A) - \delta, 0\} \\
\underline{P}(A) &= \min\{(1 + \delta)P_0(A), 1\}
\end{align*}
\]

\[ \mathcal{M}(P_0, \delta) = \{ P \text{ prob.} | \underline{P}(A) \leq P(A) \leq \overline{P}(A) \} \]
The Pari-Mutuel Model

What is known about the Pari-Mutuel Model?


- Inference and risk measurement with the pari-mutuel model, R. Pelessoni et al. IJAR, 2010.

The Pari-Mutuel Model

Take $P_0 = (0.6, 0.3, 0.1)$. 
The Pari-Mutuel Model

Take $P_0 = (0.6, 0.3, 0.1)$. 
$\delta_1 = 0.1$. 

$P_0 = \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix}$. 
$\delta_1 = 0.1$. 

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The Pari-Mutuel Model

Take $P_0 = (0.6, 0.3, 0.1)$.  
$\delta_1 = 0.1$.  
$\delta_2 = 0.2$.  

$\delta_1 = 0$.  
$\delta_2 = 0$.  

Information fusion with PMMs
The Pari-Mutuel Model

Take $P_0 = (0.6, 0.3, 0.1)$.

$\delta_1 = 0.1$.

$\delta_2 = 0.2$.

$\delta_3 = 0.5$. 

Take $P_0 = (0.6, 0.3, 0.1)$.
The Pari-Mutuel Model

Take $P_0 = (0.6, 0.3, 0.1)$.

$\delta_1 = 0.1$.
$\delta_2 = 0.2$.
$\delta_3 = 0.5$.

$\overline{P}(A) - \underline{P}(A) \leq \delta$

$\overline{P}(A), \underline{P}(A) \in (0, 1) \Rightarrow \overline{P}(A) - \underline{P}(A) = \delta$
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Probability intervals

\[ \mathcal{I} = \{ [l_i, u_i] \mid i = 1, \ldots, n \}. \]

\[ \mathcal{M}(\mathcal{I}) = \{ P \text{ prob.} \mid l_i \leq P(\{x_i\}) \leq u_i \}. \]

\( l, u \) are the lower and upper envelopes of \( \mathcal{M}(\mathcal{I}) \).

**Theorem**

Let \( P_0 \) be a probability, \( \delta > 0 \) and \( \underline{P}, \overline{P} \) the lower and upper probability induced by the PMM. Define the probability interval:

\[ \mathcal{I} = \{ [\underline{P}(\{x_i\}), \overline{P}(\{x_i\})] \mid i = 1, \ldots, n \}. \]

Then \( \mathcal{M}(\mathcal{I}) = \mathcal{M}(P_0, \delta) \), or equivalently \( \underline{P} = l \) and \( \overline{P} = u \).
Belief functions

Define \( k = \min\{|A| : \underline{P}(A) > 0\} \).

**Theorem**

Let \( \underline{P} \) be the lower probability induced by a PMM \((P_0, \delta)\). \( \underline{P} \) is a belief function if and only if one of the following conditions hold:

1. \( k = n \).

**Focal sets:** \( X, m(X) = 1 \).
Belief functions

Define $k = \min\{|A| : P(A) > 0\}$.

Theorem

Let $P$ be the lower probability induced by a PMM $(P_0, \delta)$. $P$ is a belief function if and only if one of the following conditions hold:

1. $k = n$.
2. $k = n - 1$ and $\sum_{i=1}^{n} P(X \setminus \{x_i\}) \leq 1$.

Focal sets: $X$, $X \setminus \{x\}$ $\forall x \in X$. 
Belief functions

Define $k = \min\{|A| : \underline{P}(A) > 0\}$.

**Theorem**

Let $\underline{P}$ be the lower probability induced by a PMM $(P_0, \delta)$. $\underline{P}$ is a belief function if and only if one of the following conditions hold:

1. $k = n$.
2. $k = n - 1$ and $\sum_{i=1}^{n} P(X \setminus \{x_i\}) \leq 1$.
3. $k < n - 1$, $\exists B$ with $|B| = k$ and $\underline{P}(B) > 0$, and $\underline{P}(A) > 0$ if and only if $B \subseteq A$.

**Focal sets:** $B, B \cup \{x\}, \forall x \notin B$. 
Belief functions

Define \( k = \min \{|A| : \underline{P}(A) > 0\} \).

Theorem

Let \( \underline{P} \) be the lower probability induced by a PMM \( (P_0, \delta) \). \( \underline{P} \) is a belief function if and only if one of the following conditions hold:

1. \( k = n \).
2. \( k = n - 1 \) and \( \sum_{i=1}^{n} P(X \setminus \{x_i\}) \leq 1 \).
3. \( k < n - 1 \), \( \exists! B \) with \( |B| = k \) and \( \underline{P}(B) > 0 \), and \( \underline{P}(A) > 0 \) if and only if \( B \subseteq A \).
4. \( k < n - 1 \), \( \exists! B \) with \( |B| = k - 1 \) and \( \delta = \frac{P_0(B)}{1-P_0(B)} \), and \( \underline{P}(A) > 0 \) if and only if \( B \subset A \).

Focal sets: \( B \cup \{x\}, \forall x \notin B \).
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Extreme points induced by a PMM

Since $P$ is 2-monotone, all extreme points $P_\sigma$ are generated by the permutations $\sigma$ of $\{1, \ldots, n\}$:

$$P_\sigma(\{x_{\sigma(1)}\}) = \overline{P}(\{x_{\sigma(1)}\}).$$
$$P_\sigma(\{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\}) = \overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(k)}\}).$$

Proposition

Let $P, \overline{P}$ be the lower and upper probability induced by a PMM $(P_0, \delta)$. $P_\sigma$ is given by:

$$P_\sigma(\{x_{\sigma(i)}\}) = \overline{P}(\{x_{\sigma(i)}\}) \quad \forall i = 1, \ldots, j - 1.$$
$$P_\sigma(\{x_{\sigma(j)}\}) = P(\{x_{\sigma(j)}, \ldots, x_{\sigma(n)}\}),$$
$$P_\sigma(\{x_{\sigma(j+1)}\}) = \ldots = P_\sigma(\{x_{\sigma(n)}\}) = 0,$$

where $j$ satisfies

$$\overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(j-1)}\}) < \overline{P}(\{x_{\sigma(1)}, \ldots, x_{\sigma(j)}\}) = 1.$$
Theorem

*Then maximal number of extreme points of $\mathcal{M}(P_0, \delta)$ is:*

1. $\frac{n}{2} \binom{n}{\frac{n}{2}}$, if $n$ is even.
2. $\frac{n+1}{2} \binom{n}{\frac{n+1}{2}}$, if $n$ is odd.

- It coincides with the maximal number of extreme points induced by a probability interval.
- The upper bound can be attained for the uniform distribution.
Number of extreme points

Given the PMM \((P_0, \delta)\) inducing \(\underline{P}, \overline{P}\), define:

\[
\mathcal{L} = \{ A \subseteq X \mid \overline{P}(A) = 1 \}.
\]

Proposition

Given a PMM \((P_0, \delta)\), the number of extreme points of \(M(P_0, \delta)\) is bounded above by:

\[
\sum_{A \in \mathcal{L}} \left| \bigcap_{B \subseteq A, B \in \mathcal{L}} B \right|.
\]

Furthermore, the upper bound is attained if and only if \(P_0(A) > \frac{1}{1+\delta}\) for any \(A \in \mathcal{L}\).
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

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Number of extreme points

\[ X = \{ x_1, x_2, x_3, x_4 \}, \ P_0 = (0.05, 0.15, 0.2, 0.6), \ \delta = 0.3: \]

\[
\begin{array}{c}
\{ x_1, x_2, x_3 \} \{ x_1, x_2, x_4 \} \{ x_1, x_3, x_4 \} \{ x_2, x_3, x_4 \} \\
\{ x_1, x_2 \} \{ x_1, x_3 \} \{ x_1, x_4 \} \{ x_2, x_3 \} \{ x_2, x_4 \} \\
\{ x_1 \} \{ x_2 \} \{ x_3 \} \{ x_4 \} \\
\end{array}
\]

\[
\{ x_3, x_4 \} \rightarrow \bigcap_{B \subseteq \{ x_3, x_4 \}, B \in \mathcal{L}} B = |\{ x_3, x_4 \}| = 2.
\]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

\[ \{x_1, x_2, x_3\} \quad \{x_1, x_2, x_4\} \quad \{x_1, x_3, x_4\} \quad \{x_2, x_3, x_4\} \]

\[ \{x_1, x_2\} \quad \{x_1, x_3\} \quad \{x_1, x_4\} \quad \{x_2, x_3\} \quad \{x_2, x_4\} \quad \{x_3, x_4\} \]

\[ \{x_1\} \quad \{x_2\} \quad \{x_3\} \quad \{x_4\} \]

\[ \{x_2, x_3, x_4\} \quad \rightarrow \quad \bigcap_{B \subseteq \{x_2, x_3, x_4\}, B \in \mathcal{L}} B \bigg| = |\{x_3, x_4\}| = 2. \]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

\[
\begin{align*}
\{x_1, x_2, x_3\} & \rightarrow \{x_1, x_2, x_4\} \rightarrow \{x_1, x_3, x_4\} \rightarrow \{x_2, x_3, x_4\} \\
\{x_1, x_2\} & \rightarrow \{x_1, x_3\} \rightarrow \{x_1, x_4\} \rightarrow \{x_2, x_3\} \rightarrow \{x_2, x_4\} \rightarrow \{x_3, x_4\} \\
\{x_1\} & \rightarrow \{x_2\} \rightarrow \{x_3\} \rightarrow \{x_4\}
\end{align*}
\]

\[ \left| \bigcap_{B \subseteq \{x_1, x_3, x_4\}, B \in \mathcal{L}} B \right| = |\{x_3, x_4\}| = 2. \]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

\[ X \rightarrow \left| \bigcap_{B \subseteq \{x_1, x_2, x_4\}, B \in \mathcal{L}} B \right| = |\{x_1, x_2, x_4\}| = 3. \]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \ P_0 = (0.05, 0.15, 0.2, 0.6), \ \delta = 0.3: \]

\[
X \rightarrow \bigcap_{B \subseteq X, B \in \mathcal{L}} B = |\{x_4\}| = 1.
\]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

\[ |\text{ext}(\mathcal{M}(P_0, \delta))| \leq 2 + 2 + 2 + 3 + 1. \]
Number of extreme points

\[ X = \{x_1, x_2, x_3, x_4\}, \quad P_0 = (0.05, 0.15, 0.2, 0.6), \quad \delta = 0.3: \]

\[
|\text{ext}(\mathcal{M}(P_0, \delta))| = 2 + 2 + 2 + 3 + 1.
\]
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Combining multiple PMMs

Given two PMMs \((P_0^1, \delta_1), (P_0^2, \delta_2)\), we study:

- **Conjunction:** \(\mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2)\).

- **Disjunction:** \(\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)\).

- **Mixture:** \(\varepsilon\mathcal{M}(P_0^1, \delta_1) + (1 - \varepsilon)\mathcal{M}(P_0^2, \delta_2)\).
Conjunction

Proposition
\[ \mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2) \text{ is non-empty if and only if:} \]
\[ \sum_{x \in X} \min \left\{ (1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^1(\{x\}), 1 \right\} \geq 1. \]

\[ \text{Then, it is induced by a PMM } (P_0^\cap, \delta^\cap) \text{ given by:} \]
\[ \delta^\cap = \left( \sum_{x \in X} \min \left\{ (1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^1(\{x\}), 1 \right\} \right) - 1. \]
\[ P_0^\cap = \frac{\min \left\{ (1 + \delta_1)P_0^1(\{x\}), (1 + \delta_2)P_0^1(\{x\}) \right\}}{1 + \delta^\cap}. \]
Conjunction

\[ \mathcal{M}(P_0^1, \delta_1) \cap \mathcal{M}(P_0^2, \delta_2) \]
Disjunction

Proposition

Neither $\mathcal{M}(P^1_0, \delta_1) \cup \mathcal{M}(P^2_0, \delta_2)$ nor its convex hull are induced by a PMM.

However, they can be outer-approximated by a PMM:

$$\text{conv} \left( \mathcal{M}(P^1_0, \delta_1) \cup \mathcal{M}(P^2_0, \delta_2) \right) \subseteq \mathcal{M}(P^\cup_0, \delta^\cup),$$

given by:

$$\delta^\cup = \left( \sum_{x \in X} \max \left\{ (1 + \delta_1)P^1_0(\{x\}), (1 + \delta_2)P^2_0(\{x\}) \right\} \right) - 1.$$

$$P^\cup_0 = \frac{\max \left\{ (1 + \delta_1)P^1_0(\{x\}), (1 + \delta_2)P^2_0(\{x\}) \right\}}{1 + \delta^\cup}.$$
Disjunction

\[ \mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2) \]
Disjunction

\[ \text{conv}(\mathcal{M}(P_0^1, \delta_1) \cup \mathcal{M}(P_0^2, \delta_2)) \]
Disjunction

\[ \mathcal{M}(P_0^\cup, \delta^\cup) \]
Mixture

Proposition

$\varepsilon \mathcal{M}(P_0^1, \delta_1) + (1 - \varepsilon) \mathcal{M}(P_0^2, \delta_2)$ is induced by a PMM $(P_0^\varepsilon, \delta^\varepsilon)$ given by:

\[
\delta^\varepsilon = \varepsilon (1 + \delta_1) + (1 - \varepsilon)(1 + \delta_2) - 1.
\]

\[
P_0^\varepsilon = \frac{\varepsilon (1 + \delta_1) P_0^1(\{x\}) + (1 - \varepsilon)(1 + \delta_2) P_0^2(\{x\})}{1 + \delta^\varepsilon}.
\]
Mixture

\[ \epsilon \mathcal{M}(P_0^1, \delta_1) + (1 - \epsilon)\mathcal{M}(P_0^2, \delta_2) \]
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The PMM as an imprecise probability model:

- Extension to gambles (*Pelessoni et al.*, *Walley*).
- The PMM and risk measures (*Pelessoni et al.*).
- Conditioning a PMM (*Pelessoni et al.*).
- PMM with a uniform distribution (*Utkin*).
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The PMM as an imprecise probability model:
- Extension to gambles (*Pelessoni et al.*, *Walley*).
- The PMM and risk measures (*Pelessoni et al.*).
- Conditioning a PMM (*Pelessoni et al.*).
- PMM with a uniform distribution (*Utkin*).
- Connection with other models of the IP Theory.
- Extreme points of $\mathcal{M}(P_0, \delta)$.
- Merging information given in terms of PMMs.
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