A Semantics for Conditionals with Default Negation

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July 13

ECSQARU 2017

Motivating Example

Cars

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from The Noun Project

Motivating Example

- *Cars* typically have a *gasoline* engine.

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- *Cars* typically have a *gasoline* engine.
- *Electric* driven cars typically have an *electric* engine instead.

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- *Hybrid* cars may feature *both* engine types, but don't necessarily have to.

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- *Cars* typically have a *gasoline* engine.
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$\Rightarrow$ Would you expect a hybrid car to have a gasoline engine or not?

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⇒ Would you expect a hybrid car to have a gasoline engine or not?

Given the above information,
we'd like to have an epistemic state which is indifferent about this.

Formal Conditional Assertions

$A \Rightarrow B$	Material Implication	"If A holds, then B" (*No exceptions*)
$B \leftarrow A, \text{not } C$	Default Negation	"If A holds, and C is not provable, then B" (*Explicit exceptions*)
$(B\|A)$	Conditional	"If A holds, then usually B" (*Implicit exceptions*)

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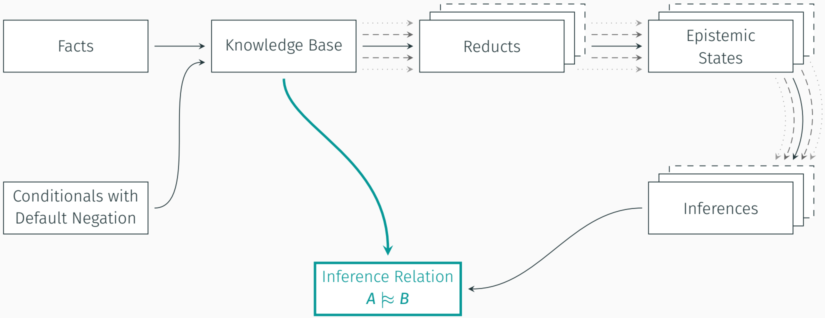
$(B\|A, \text{not } C)$	Conditional	"If A holds, and C is not provable, then usually B" (*Both types of exceptions*)

Overview



Facts
Knowledge Base
Reducts
Epistemic States
Conditionals with Default Negation
Inferences
Inference Relation
$A \approx B$

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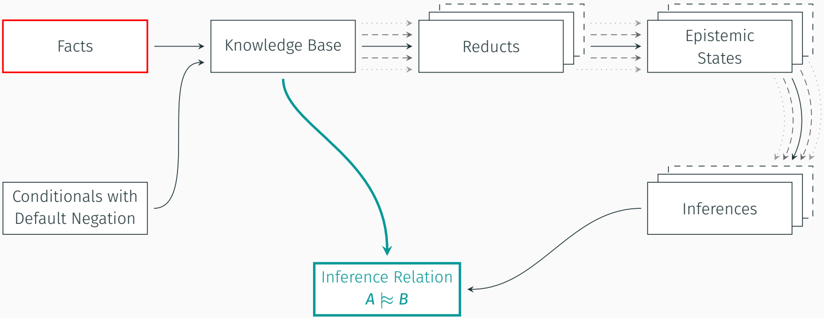

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Logic and Worlds

We use a ***standard propositional logic*** with

- A finite propositional alphabet $\Sigma = \{V_1, \ldots, V_m\}$,
- The usual logical connectives $\wedge, \vee, \neg$, and
- A language $\mathfrak{L}$ of literals from Σ closed under these connectives.

We represent the set of ***possible worlds*** Ω syntactically with complete conjunctions of literals of Σ.

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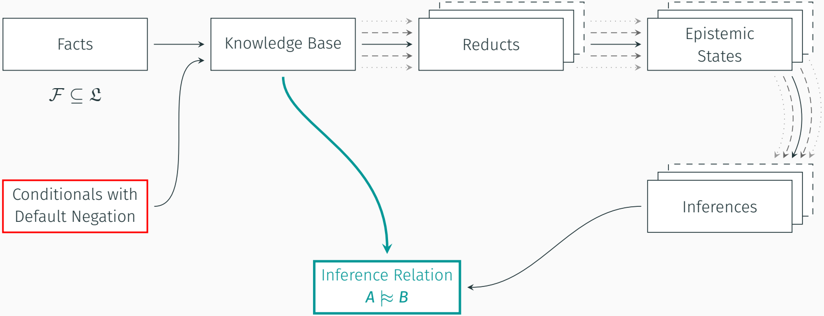
We represent the set of ***possible worlds*** Ω syntactically with complete conjunctions of literals of Σ.

Example (Possible Worlds)

Let $\Sigma = \{E, G, H\}$ be the alphabet of our running example of ***e***lectric and ***g***asoline engines and ***h***ybrid cars.

The possible worlds for this alphabet are:

$$\Omega = \left\{ egh, eg\overline{h}, e\overline{g}h, e\overline{g}\overline{h}, \overline{e}gh, \overline{e}g\overline{h}, \overline{e}\overline{g}h, \overline{e}\overline{g}\overline{h} \right\}.$$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Conditionals

- Conditionals $(B|A)$ encode defeasible rules *"If A then usually B"*.
- Three-valued evaluation by worlds [Fin74]:

$$[\![(B|A)]\!]_\omega = \begin{cases} \mathit{true} & \text{iff } \omega \models AB & (\text{"Rule verified"}) \\ \mathit{false} & \text{iff } \omega \models A\overline{B} & (\text{"Rule violated"}) \\ \mathit{undefined} & \text{iff } \omega \models \overline{A} & (\text{"Rule not applicable"}) \end{cases}$$

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Example (Formalizing the Introductory Example)

$(e \mid h)$: "Hybrids usually have an electric engine."

$(g\,h \mid e(g\,h \vee \overline{g}\,\overline{h}))$: "Hybrid cars with both an electric and a gasoline engine are more prominent than non-hybrids with only an electric engine."

Definition

Let $A, B \in \mathfrak{L}$ be formulas, let $\mathcal{D} \subseteq \mathfrak{L}$ be a set of formulas.
$(B \,|\, A, \text{not } \mathcal{D})$ is a ***conditional with default negation***.

If $\mathcal{D} = \emptyset$, we write $(B \,|\, A)$ instead of $(B \,|\, A, \text{not } \emptyset)$.

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$(\overline{g} \,|\, e \qquad)$: "Electric cars typically don't have a gasoline engine."

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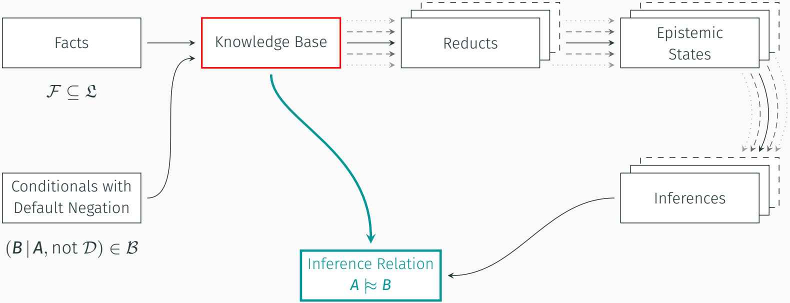
$(g\,h \,|\, e(g\,h \vee \overline{g}\,\overline{h}))$: "Hybrid cars with both an electric and a gasoline engine are more prominent than non-hybrids with only an electric engine."
$(\overline{g} \,|\, e, \text{not } \{h\})$: "Electric cars typically don't have a gasoline engine — unless they are hybrids."

Overview



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Definition

A ***knowlegde base*** $\mathcal{R}$ is comprised of

- a set of formulas $\mathcal{F}_{\mathcal{R}}$ (***facts***) and
- a set of conditionals with default negation $\mathcal{B}_{\mathcal{R}}$ (***beliefs***).

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Example (Formalizing the Introductory Example (contd.))

Facts:	$h \Rightarrow e$	Hybrids are cars with an electric engine.
Beliefs:	$(\overline{g} \mid e, \text{not } \{h\})$	Electric cars typically don't have a gasoline engine — unless they are hybrids.
	$(g\,h \mid e(g\,h \vee \overline{g}\,\overline{h}))$	Cars with electric engines are more likely to be hybrids with a gasoline engine than non-hybrids without.

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	$(g\,h \mid e(g\,h \vee \overline{g}\,\overline{h}))$	Cars with electric engines are more likely to be hybrids with a gasoline engine than non-hybrids without.

Together, we have

$$\mathcal{R} = \left\{ \underbrace{\{h \Rightarrow e\}}_{\mathcal{F}}, \underbrace{\{(\overline{g} \mid e, \text{not } \{h\}), (g\,h \mid e(g\,h \vee \overline{g}\,\overline{h}))\}}_{\mathcal{B}} \right\}$$

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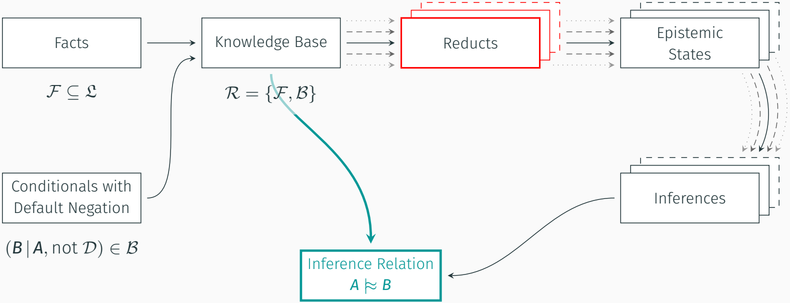

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Reduct

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The ***reduct*** $\mathcal{R}^S = (\mathcal{F}_\mathcal{R}, \mathcal{B}_\mathcal{R}^S)$ of $\mathcal{R}$ by some formula $S \in \mathfrak{L}$ is the knowledge base $\mathcal{R}$ with its set of beliefs $\mathcal{B}_\mathcal{R}$ being replaced by

$$\mathcal{B}_\mathcal{R}^S = \big\{(\boldsymbol{B}\,|\,\boldsymbol{A}) \mid (\boldsymbol{B}\,|\,\boldsymbol{A}, \text{not}\,\mathcal{D}) \in \mathcal{B}_\mathcal{R} \quad \text{and} \quad \forall\, \boldsymbol{D} \in \mathcal{D} : \{\boldsymbol{S}\} \cup \mathcal{F}_\mathcal{R} \not\models \boldsymbol{D}\big\}.$$

Definition

The ***reduct*** $\mathcal{R}^S = (\mathcal{F}_\mathcal{R}, \mathcal{B}^S_\mathcal{R})$ of $\mathcal{R}$ by some formula $S \in \mathfrak{L}$ is the knowledge base $\mathcal{R}$ with its set of beliefs $\mathcal{B}_\mathcal{R}$ being replaced by

$$\mathcal{B}^S_\mathcal{R} = \big\{(B \,|\, A) \mid (B \,|\, A, \text{not } \mathcal{D}) \in \mathcal{B}_\mathcal{R} \quad \text{and} \quad \forall\, D \in \mathcal{D} : \{S\} \cup \mathcal{F}_\mathcal{R} \not\models D\big\}.$$

Example (Reducts in the Car Example)

For the knowledge base
$\mathcal{R} = \Big\{\{h \Rightarrow e\}, \{(\overline{g} \,|\, e, \text{not } \{h\}), (g\,h \,|\, e(g\,h \vee \overline{g}\,\overline{h}))\}\Big\}$

we have the reducts

$$\mathcal{B}^S = \left\{ \begin{array}{c} \sout{(\overline{g} \,|\, e, \text{not } \{h\})} \\ (g\,h \,|\, e(g\,h \vee \overline{g}\,\overline{h})) \end{array} \right\} \quad \text{and} \quad \mathcal{B}^{S'} = \left\{ \begin{array}{c} (\overline{g} \,|\, e, \sout{\text{not } \{h\}}), \\ (g\,h \,|\, e(g\,h \vee \overline{g}\,\overline{h})) \end{array} \right\}$$

for any formulas S with $S \models h$ and S' with $S' \not\models h$.

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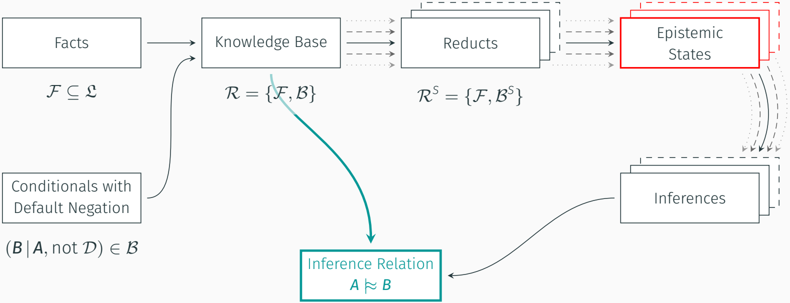

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An Ordinal Conditional Function (OCF) or ***ranking function*** κ is a function that assigns a ***degree of disbelief*** to each world $\omega \in \Omega$.

Definition (OCF [Spo88])

$\kappa := \Omega \to \mathbb{N}_0^\infty$ such that:

$$\kappa^{-1}(0) \neq \emptyset$$

$$\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$$

$$\kappa(B \mid A) = \kappa(A\,B) - \kappa(A)$$

$$\kappa \models (B \mid A) \text{ iff } \kappa(A\,B) < \kappa(A\,\overline{B})$$

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Example (Car Ranking)

$\overline{e} g h$, $\overline{e} \overline{g} h$	$\kappa(\omega) = \infty$
$e \overline{g} \overline{h}$	$\kappa(\omega) = 2$
$e g h$, $e g \overline{h}$	$\kappa(\omega) = 1$
$e \overline{g} h$, $\overline{e} g \overline{h}$, $\overline{e} \overline{g} \overline{h}$	$\kappa(\omega) = 0$

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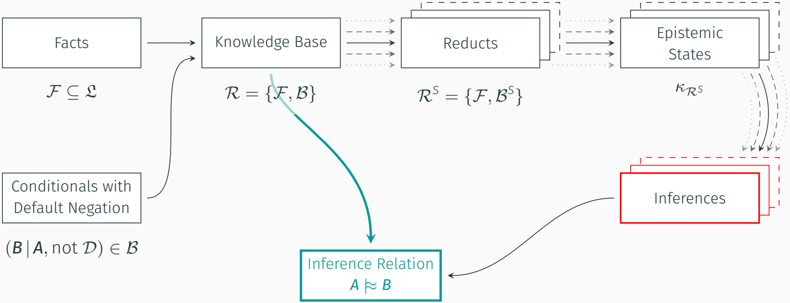

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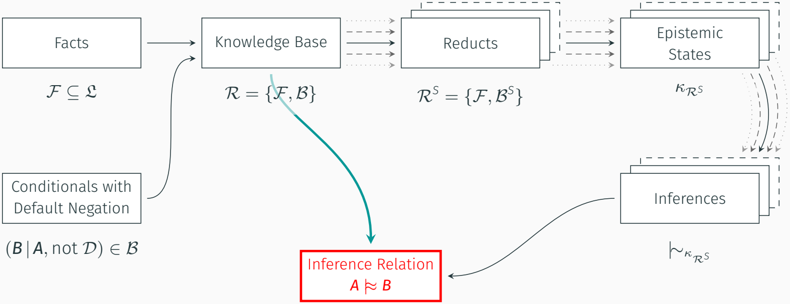
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Definition

A infers B in the context of a knowledge base $\mathcal{R}$ with conditionals with default negation iff A infers B in the epistemic state of the reduct $\mathcal{R}^A$: $\quad A \mathrel{|\!\!\approx} B \qquad \text{iff} \qquad A \mathrel{|\!\!\sim}_{\kappa_{\mathcal{R}^A}} B.$

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$\kappa(\omega)$	
∞	$\overline{e}\,g\,h,\ \overline{e}\,\overline{g}\,h$
1	$e\,\overline{g}\,\overline{h}$
0	$e\,g\,h, e\,g\,\overline{h},$ $e\,\overline{g}\,h,\ \overline{e}\,g\,\overline{h},\ \overline{e}\,\overline{g}\,\overline{h}$

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A infers B in the context of a knowledge base $\mathcal{R}$ with conditionals with default negation iff A infers B in the epistemic state of the reduct $\mathcal{R}^A$: $\quad A \approx B \qquad \text{iff} \qquad A \mid\!\sim_{\kappa_{\mathcal{R}^A}} B$.

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0	$e\,g\,h$, $e\,g\,\overline{h}$, $e\,\overline{g}\,h$, $\overline{e}\,g\,\overline{h}$, $\overline{e}\,\overline{g}\,\overline{h}$

$\Rightarrow$ Hybrid cars ***may or may not*** have a gasoline engine; they are ***indifferent*** towards a property of their superclass!

$h \not\approx g$ and $h \not\approx \overline{g}$

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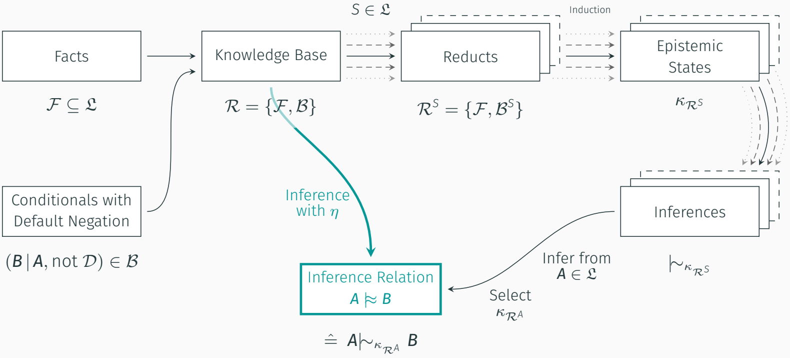

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Inference with η
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Infer from $A \in \mathfrak{L}$
Select $\kappa_{\mathcal{R}^A}$
Inference Relation
$A \approx B$
$\hat{=}\; A \mid\!\sim_{\kappa_{\mathcal{R}^A}} B$

Properties of the Inference Relation

The inference relation $\approx$ satisfies the following formal properties:

(LLE)	$A \equiv B$	and	$A \mid\!\approx C$	imply	$B \mid\!\approx C$
(RW)	$B \models C$	and	$A \mid\!\approx B$	imply	$A \mid\!\approx C$
(AND)	$A \mid\!\approx B$	and	$A \mid\!\approx C$	imply	$A \mid\!\approx BC$
(MPC)	$A \mid\!\approx B$	and	$A \mid\!\approx B \Rightarrow C$	imply	$A \mid\!\approx C$

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However, the relation neither satisfies (CUT) nor (CM) in general.

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(MPC)	$A \approx B$ and $A \approx B \Rightarrow C$	imply	$A \approx C$

However, the relation neither satisfies (CUT) nor (CM) in general. These can be satisfied under certain restrictions, however:

(CM)	$A \approx B$ and $A \approx C$	imply	$AB \approx C$
(CUT)	$A \approx B$ and $AB \approx C$	imply	$A \approx C$

given $\mathcal{R}^A = \mathcal{R}^{AB}$

Conclusion

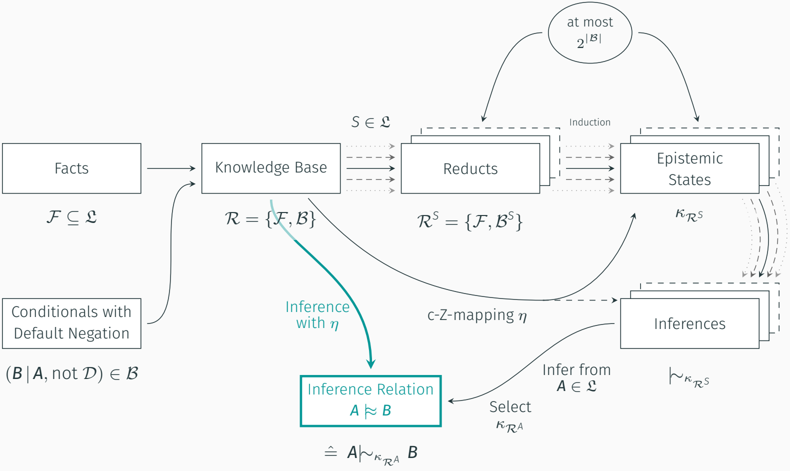
We...

... introduced ***default negation*** (known from answer set programming) into ***conditionals***.
$\rightarrow$ Proper expansion of the conditional language

... defined a novel ***inference relation*** on top of these conditionals
$\rightarrow$ Formal properties in the paper.

... are now capable of modeling exceptions such as subclass indifference (e.g., hybrid cars).

The Whole Paper on One Slide



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