Generalized Probabilistic Modus Ponens [1]

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Abstract

- Modus ponens (*from A and "if A then C" infer C*) is one of the most basic inference rules.
- The probabilistic modus ponens allows for managing uncertainty by transmitting assigned uncertainties from the premises to the conclusion (i.e., *from* $P(A)$ *and* $P(C|A)$ *infer* $P(C)$).
- We generalize the probabilistic modus ponens by replacing A by the conditional event $A|H$.
- The resulting inference rule involves iterated conditionals (formalized by conditional random quantities) and propagates previsions from the premises to the conclusion.
- Interestingly, the propagation rules for the lower and the upper bounds on the conclusion of the generalized probabilistic modus ponens coincide with the respective bounds on the conclusion of the (non-nested) probabilistic modus ponens.

Modus ponens

Modus ponens	Instantiation
A	The son gets a B
If A, then C	If the son gets a B, then the mother is angry
C	The mother is angry

Probabilistic MP

We recall that, given two logically independent events A and C, the set of all coherent assessment (x, y) on $\{A, C|A\}$ is the unit square $[0, 1]^2$.

	Modus Ponens	Probabilistic Modus Ponens
(Categorical prem.)	A	$P(A) = x$
(Conditional prem.)	If A, then C	$P(C|A) = y$
(Conclusion)	C	$xy \le P(C) \le xy + 1 - x$

That is, the set of all coherent assessment (x, y, z) on $\{A, C|A, C\}$ is

$$\{(x, y, z) \in [0, 1]^3 : (x, y) \in [0, 1]^2, xy \le z \le xy + 1 - x\}$$

From Modus ponens to Generalized modus ponens

	Modus ponens	Generalized modus ponens
(Categorical premise)	A	$A\|H$
(Conditional premise)	If A, then C	If $A\|H$, then C
(Conclusion)	C	C

Instantiation (Gibbard, 1981, p. 237)

$\overbrace{\textit{The cup breaks if dropped}}^{A|H}$.

If $\overbrace{\textit{the cup breaks if dropped}}^{A|H}$, then $\overbrace{\textit{the cup is fragile}}^{C}$.

Therefore, $\overbrace{\textit{the cup is fragile}}^{C}$.

Generalized Probabilistic MP

We will generalize the probabilistic modus ponens by replacing the categorical premise (i.e., A) and the antecedent of the conditional premise (i.e., A in "*if A then C*") by the conditional event $A|H$.

Generalized Modus Ponens	Generalized Probabilistic Modus Ponens
$A\|H$	$P(A\|H) = x$
If $A\|H$, then C	$\mathbb{P}(C\|(A\|H)) = y$ (What is $C\|(A\|H)$?)
C	$P(C) \in [?, ?]$

What does the conditional premise (i.e., an iterated conditional) mean and how can we assess its uncertainty?
What is the set of all coherent assessment (x, y) on $\{A|H, C|(A|H)\}$?
What is the set of all coherent assessment (x, y, z) on $\{A|H, C|(A|H), C\}$?

Cond. probabilities as probabilities of conditional event

By using the same symbols to denote events and their indicators, agreeing to the betting metaphor of the coherence framework, if you assess $p = P(A|H)$, then coherence requires that $p = \mathbb{P}(AH + p\overline{H})$. Thus, we identify the conditional event $A|H$ as the following random quantity

(see, e.g., Lad, 1996; Gilio & Sanfilippo, 2014)

$$A|H = AH + p\overline{H} = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ p, & \text{if } \overline{H} \text{ is true,} \end{cases}$$

Given a random quantity X and an event H, we identify

$$X|H = XH + \mu\overline{H}, \text{ where } \mu = \mathbb{P}(X|H).$$

Conjoined conditionals

Definition (Conjunction)

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x, P(B|K) = y$, we define their *conjunction* as the following conditional random quantity (see, e.g., Gilio & Sanfilippo, 2013a, 2013b, 2014)

$$(A|H) \wedge (B|K) = min\{A|H, B|K\} \,|\, (H \vee K) = min\{AH + x\overline{H}, BK + y\overline{K}\} \,|\, (H \vee K)\,.$$

Of course $(A|H) \wedge (B|H) = (A \wedge B)|H$.
Notice that, if $x = y = 1$, then the conditional events $A|H = AH + x\overline{H}, B|K = BK + y\overline{K}$ coincide with the material conditionals $AH + \overline{H}, BK + \overline{K}$ and we recovery the quasi conjunction of Adams:

$$min\{AH + \overline{H}, BK + \overline{K}\} \,|\, (H \vee K) = QC(A|H, B|K)$$

Interpretation by the betting scheme

By assessing $\mathbb{P}[(A|H) \wedge (B|K)] = z$, you agree to pay z with the proviso that you will receive:

$$(A|H) \wedge (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true or } \overline{B}K \text{ is true,} \\ x = P(A|H), & \text{if } \overline{H}BK \text{ is true,} \\ y = P(B|K), & \text{if } \overline{K}AH \text{ is true,} \\ z = \mathbb{P}[(A|H) \wedge (B|K)], & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases}$$

In other words, you will receive:

- 1, if both conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if a conditional event is void and the other one is true;
- the quantity that you paid, if both conditional events are void.

Fréchet-Hoeffding bounds (Gilio & Sanfilippo, 2014): z is coherent iff:

$$max\{x + y - 1, 0\} \leq z \leq min\{x, y\}. \tag{1}$$

Quasi-Conjunction and Conjunction

Let A, H, B, K be log. ind. events, with $H \neq \bot, K \neq \bot$. Assuming that $x = P(A|H)$ and $y = P(B|K)$, we obtain

		$A\|H$	$B\|K$	$\mathcal{QC}(A\|H, B\|K)$	$A\|H \wedge B\|K$
$C_1 = AHBK$	$\Rightarrow$	1	1	1	1
$C_2 = AHB^cK$	$\Rightarrow$	1	0	0	0
$C_3 = A^cHBK$	$\Rightarrow$	0	1	0	0
$C_4 = A^cHB^cK$	$\Rightarrow$	0	0	0	0
$C_5 = H^cBK$	$\Rightarrow$	x	1	1	x
$C_6 = AHK^c$	$\Rightarrow$	1	y	1	y
$C_7 = A^cHK^c$	$\Rightarrow$	0	y	0	0
$C_8 = H^cB^cK$	$\Rightarrow$	x	0	0	0
$C_9 = H^cK^c$	$\Rightarrow$	x	y	$\nu = \mathbb{P}(QC(A\|H, B\|K))$	$z = \mathbb{P}(A\|H \wedge B\|K)$

QC bounds: ν is coherent iff $max\{x+y-1, 0\} \leq \nu \leq S_0^H(x,y)$

where $S_0^H(x,y) = \begin{cases} \frac{x-y-2xy}{1-xy}, & (x,y) \neq (1,1), \\ 1, & (x,y) = (1,1) \end{cases}$

Iterated conditional and betting scheme

Definition

The iterated conditional $(B|K)|(A|H)$ is the conditional random quantity

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu \cdot \overline{A}|H, \qquad (2)$$

where $\mu = \mathbb{P}[(B|K)|(A|H)]$. Notice that (2) is a generalization of $A|H = A \wedge H + p \cdot \overline{H}$, where $p = P(A|H)$.

In the context of betting scheme μ represents the amount you agree to pay with the proviso that you will receive the quantity

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ true,} \\ 0, & \text{if } AH\overline{B}K \text{ true,} \\ y, & \text{if } AH\overline{K} \text{ true,} \\ \mu, & \text{if } \overline{A}H \text{ true,} \\ x + \mu(1-x), & \text{if } \overline{H}BK \text{ true,} \\ \mu(1-x), & \text{if } \overline{H}\overline{B}K \text{ true,} \\ z + \mu(1-x), & \text{if } \overline{H}\overline{K} \text{ true.} \end{cases}$$

Coherence requires that $z + \mu(1-x) = \mu \in [0,1]$.

Some properties

- The product rule: (Gilio & Sanfilippo, 2014)

$$\mathbb{P}[(B|K) \wedge (A|H)] = \mathbb{P}[(B|K)|(A|H)] \cdot P(A|H)\,. \qquad (3)$$

Moreover, assuming $x = P(A|H) > 0$, one has:

$$\mathbb{P}[(B|K)|(A|H)] = \mu = \frac{\mathbb{P}[(B|K) \wedge (A|H)]}{P(A|H)} = \frac{z}{x}\,.$$

- Decomposition formula

$$B|K = (A|H) \wedge (B|K) + (\bar{A}|H) \wedge (B|K)\,. \qquad (4)$$

By the linearity of prevision, and by the product rule, we obtain

$$P(B|K) = \mathbb{P}[(B|K)|(A|H)]P(A|H) + \mathbb{P}[(B|K)|(\bar{A}|H)]P(\bar{A}|H).$$

A particular case

If $K = \Omega$ (by replacing B with C), then we obtain from the decomposition formula:

$$C = (A|H) \wedge C + (\overline{A}|H) \wedge C\,, \tag{5}$$

$$P(C) = \mathbb{P}[C|(A|H)]P(A|H) + \mathbb{P}[C|(\overline{A}|H)]P(\overline{A}|H) \tag{6}$$

By applying Definition 2, with $K = \Omega$ and by replacing B with C, we obtain $C|(A|H) = C \wedge (A|H) + \mu\overline{A}|H$. That is,

$$C|(A|H) = \begin{cases} 1, & \text{if } AHC \text{ true}, \\ 0, & \text{if } AH\overline{C} \text{ true}, \\ \mu, & \text{if } \overline{A}H \text{ true}, \\ x + \mu(1 - x), & \text{if } \overline{H}C \text{ true}, \\ \mu(1 - x), & \text{if } \overline{H}\overline{C} \text{ true}, \end{cases}$$

Notice that $C|(A|H) \neq C|AH$.

Coherent sets and coherent extensions

- **Theorem (2)**

  Let three logically independent events A, C, H be given, with $A \neq \bot$, $H \neq \bot$. The set of all coherent assessments $\mathcal{M} = (x, y, z)$ on $\mathcal{F} = \{A|H, C|(A|H), C|(\overline{A}|H)\}$ is the unit cube $[0,1]^3$.

  That is, assuming logical independence, every point $(x, y, z) \in [0,1]^3$ is a coherent assessment on $\{A|H, C|(A|H), C|(\overline{A}|H)\}$.

- **Theorem (3)**

  Given any coherent assessment (x, y) on $\{A|H, C|(A|H)\}$, with A, C, H logically independent, but $A \neq \bot$ and $H \neq \bot$, the extension $z = P(C)$ is coherent if and only if $z \in [z', z'']$, where

$$z' = xy \quad \textit{and} \quad z'' = xy + 1 - x \,. \tag{7}$$

That is, we generalized the probabilistic *modus ponens* to the case where the first premise A is replaced by the conditional event $A|H$.

Proof

From $\boxed{P(A|H) = x}$ and $\boxed{\mathbb{P}[C|(A|H)] = y}$ infer $\boxed{xy \leq P(C) \leq xy + 1 - x}$
From (6), by the linearity of prevision, and the product rule, we obtain

$$z = P(C) = \underbrace{P(A|H)}_{x}\,\underbrace{\mathbb{P}[C|(A|H)]}_{y} + \underbrace{P(\bar{A}|H)}_{1-x}\,\underbrace{\mathbb{P}[C|(\bar{A}|H)]}_{t\in[0,1]}$$

From Theorem 3, given any coherent assessment (x, y) on $\{A|H, C|(A|H)\}$, the extension $t = \mathbb{P}[C|(\bar{A}|H)]$ on $C|(\bar{A}|H)$ is coherent for every $t \in [0, 1]$. As $z = xy + (1 - x)t$, it follows that

$$\underbrace{xy}_{if\ \ t=0} \leq P(C) \leq \underbrace{xy + (1 - x)}_{if\ \ t=1}$$

Prevision entailment:

$$P(A|H) = 1,\ \&\ \mathbb{P}[C|(A|H)] = 1 \Longrightarrow P(C) = 1$$

Conclusion

- We generalized the probabilistic modus ponens in terms of conditional random quantities in the setting of coherence.
- Specifically, we replaced the categorical premise A and the antecedent A of the conditional premise $C|A$ by the conditional event $A|H$.
- We proved a generalized decomposition formula for conditional events and we gave some results.
- We propagated the previsions from the premises of the generalized probabilistic modus ponens to the conclusion.
- We have shown that the lower and the upper bounds on the conclusion of the generalized probabilistic modus ponens coincide with the respective bounds on the conclusion for the (non-nested) probabilistic modus ponens.

Further work

- We will study other instantiations to obtain further generalizations of modus ponens, e.g., by also replacing the consequent C of the conditional premise $C|A$ and the conclusion C by a conditional event $C|K$: from $\{A|H, (C|K)|(A|H)\}$ infer $C|K$.
- We will focus on similar generalizations (also involving imprecision) of other argument forms like the probabilistic modus tollens.
- We will study similar generalization of inference rules in System P where a conditional will be replaced by an iterated one.

Thank you for your attention!

Link to the full paper

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Some references I

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