Generalized Probabilistic Modus Ponens

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Abstract

- Modus ponens (from $A$ and “if $A$ then $C$” infer $C$) is one of the most basic inference rules.
- The probabilistic modus ponens allows for managing uncertainty by transmitting assigned uncertainties from the premises to the conclusion (i.e., from $P(A)$ and $P(C|A)$ infer $P(C)$).
- We generalize the probabilistic modus ponens by replacing $A$ by the conditional event $A|H$.
- The resulting inference rule involves iterated conditionals (formalized by conditional random quantities) and propagates previsions from the premises to the conclusion.
- Interestingly, the propagation rules for the lower and the upper bounds on the conclusion of the generalized probabilistic modus ponens coincide with the respective bounds on the conclusion of the (non-nested) probabilistic modus ponens.
Modus ponens

<table>
<thead>
<tr>
<th>Modus ponens</th>
<th>Instantiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>The son gets a B</td>
</tr>
<tr>
<td>If $A$, then $C$</td>
<td>If the son gets a B, then the mother is angry</td>
</tr>
<tr>
<td>$C$</td>
<td>The mother is angry</td>
</tr>
</tbody>
</table>
Probabilistic MP

We recall that, given two logically independent events $A$ and $C$, the set of all coherent assessment $(x, y)$ on $\{A, C|A\}$ is the unit square $[0, 1]^2$.

<table>
<thead>
<tr>
<th>(Categorical prem.)</th>
<th>Modus Ponens</th>
<th>Probabilistic Modus Ponens</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$P(A) = x$</td>
<td></td>
</tr>
<tr>
<td>If $A$, then $C$</td>
<td>$P(C</td>
<td>A) = y$</td>
</tr>
<tr>
<td>$C$</td>
<td>$xy \leq P(C) \leq xy + 1 - x$</td>
<td></td>
</tr>
</tbody>
</table>

That is, the set of all coherent assessment $(x, y, z)$ on $\{A, C|A, C\}$ is

$$\{(x, y, z) \in [0, 1]^3 : (x, y) \in [0, 1]^2, xy \leq z \leq xy + 1 - x\}$$
From Modus ponens to Generalized modus ponens

<table>
<thead>
<tr>
<th>Categorical premise</th>
<th>Modus ponens</th>
<th>Generalized modus ponens</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A(\mid H)</td>
</tr>
<tr>
<td>If A, then C</td>
<td>C</td>
<td>If A(\mid H), then C</td>
</tr>
<tr>
<td>Conclusion</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

Instantiation (Gibbard, 1981, p. 237)

\[
A\mid H
\]

*The cup breaks if dropped.*

\[
A\mid H
\]

If *the cup breaks if dropped*, then *the cup is fragile.*

\[
C
\]

Therefore, *the cup is fragile.*
Generalized Probabilistic MP

We will generalize the probabilistic modus ponens by replacing the categorical premise (i.e., \(A\)) and the antecedent of the conditional premise (i.e., \(A \) in “if \(A\) then \(C\)”) by the conditional event \(A|H\).

<table>
<thead>
<tr>
<th>Generalized Modus Ponens</th>
<th>Generalized Probabilistic Modus Ponens</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A</td>
<td>H)</td>
</tr>
<tr>
<td>If (A</td>
<td>H), then (C)</td>
</tr>
<tr>
<td>(C)</td>
<td>(P(C) \in [?, ?])</td>
</tr>
</tbody>
</table>

What does the conditional premise (i.e., an iterated conditional) mean and how can we assess its uncertainty?

What is the set of all coherent assessment \((x, y)\) on \(\{A|H, C|(A|H)\}\)?

What is the set of all coherent assessment \((x, y, z)\) on \(\{A|H, C|(A|H), C\}\)?
Cond. probabilities as probabilities of conditional event

By using the same symbols to denote events and their indicators, agreeing to the betting metaphor of the coherence framework, if you assess \( p = P(A|H) \), then coherence requires that \( p = \mathbb{P}(AH + pH) \). Thus, we identify the conditional event \( A|H \) as the following random quantity (see, e.g., Lad, 1996; Gilio & Sanfilippo, 2014)

\[
A|H = AH + pH = \begin{cases} 
1, & \text{if } AH \text{ is true}, \\
0, & \text{if } \overline{A}H \text{ is true}, \\
p, & \text{if } \overline{H} \text{ is true},
\end{cases}
\]

Given a random quantity \( X \) and an event \( H \), we identify

\[
X|H = XH + \mu \overline{H}, \quad \text{where } \mu = \mathbb{P}(X|H).
\]
Conjoined conditionals

Definition (Conjunction)

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as the following conditional random quantity (see, e.g., Gilio & Sanfilippo, 2013a, 2013b, 2014)

$$(A|H) \land (B|K) = \min\{A|H, B|K\} | (H \lor K) = \min\{AH + xH, BK + yK\} | (H \lor K).$$

Of course $(A|H) \land (B|H) = (A \land B)|H$.

Notice that, if $x = y = 1$, then the conditional events $A|H = AH + xH, B|K = BK + yK$ coincide with the material conditionals $AH + H, BK + K$ and we recovery the quasi conjunction of Adams:

$$\min\{AH + H, BK + K\} | (H \lor K) = QC(A|H, B|K)$$
Interpretation by the betting scheme

By assessing $\mathbb{P}[(A|H) \land (B|K)] = z$, you agree to pay $z$ with the proviso that you will receive:

$$(A|H) \land (B|K) = \begin{cases} 
1, & \text{if } AHBK \text{ is true,} \\
0, & \text{if } \overline{A}H \text{ is true or } \overline{B}K \text{ is true,} \\
x = P(A|H), & \text{if } \overline{H}BK \text{ is true,} \\
y = P(B|K), & \text{if } \overline{K}AH \text{ is true,} \\
z = \mathbb{P}[(A|H) \land (B|K)], & \text{if } \overline{H}\overline{K} \text{ is true.}
\end{cases}$$

In other words, you will receive:

- 1, if both conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if a conditional event is void and the other one is true;
- the quantity that you paid, if both conditional events are void.

Fréchet-Hoeffding bounds (Gilio & Sanfilippo, 2014): $z$ is coherent iff:

$$\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}. \quad (1)$$
Quasi-Conjunction and Conjunction

Let $A, H, B, K$ be log. ind. events, with $H \neq \bot, K \neq \bot$. Assuming that $x = P(A|H)$ and $y = P(B|K)$, we obtain

|               | $A|H$ | $B|K$ | $QC(A|H, B|K)$ | $A|H \land B|K$ |
|---------------|------|------|----------------|----------------|
| $C_1 = AHBK$  | 1    | 1    | 1              | 1              |
| $C_2 = AHB^cK$| 1    | 0    | 0              | 0              |
| $C_3 = A^cHBK$| 0    | 1    | 0              | 0              |
| $C_4 = A^cHB^cK$| 0  | 0    | 0              | 0              |
| $C_5 = H^cBK$ | $x$  | 1    | 1              | $x$            |
| $C_6 = AHK^c$ | 1    | $y$  | 1              | $y$            |
| $C_7 = A^cHK^c$| 0  | $y$  | 0              | 0              |
| $C_8 = H^cB^cK$| $x$| 0    | 0              | 0              |
| $C_9 = H^cK^c$ | $x$ | $y$  | $\nu = P(QC(A|H, B|K))$ | $z = P(A|H \land B|K)$ |

QC bounds: $\nu$ is coherent iff $\max\{x + y - 1, 0\} \leq \nu \leq S_0^H(x, y)$

where $S_0^H(x, y) = \begin{cases} \frac{x-y-2xy}{1-xy}, & (x, y) \neq (1, 1), \\ 1, & (x, y) = (1, 1) \end{cases}$
Iterated conditional and betting scheme

**Definition**
The iterated conditional $(B|K)|(A|H)$ is the conditional random quantity

$$(B|K)|(A|H) = (B|K) \land (A|H) + \mu \cdot \overline{A}|H, \quad (2)$$

where $\mu = \mathbb{P}[(B|K)|(A|H)]$. Notice that (2) is a generalization of $A|H = A \land H + p \cdot \overline{H}$, where $p = P(A|H)$.

In the context of betting scheme $\mu$ represents the amount you agree to pay with the proviso that you will receive the quantity

$$(B|K)|(A|H) = \begin{cases} 
1, & \text{if } AHBK \text{ true,} \\
0, & \text{if } AH\overline{B}K \text{ true,} \\
y, & \text{if } AH\overline{K} \text{ true,} \\
\mu, & \text{if } \overline{A}H \text{ true,} \\
x + \mu(1-x), & \text{if } \overline{HB}K \text{ true,} \\
\mu(1-x), & \text{if } \overline{HB}K \text{ true,} \\
z + \mu(1-x), & \text{if } \overline{HK} \text{ true.}
\end{cases}$$

Coherence requires that $z + \mu(1-x) = \mu \in [0,1]$. 
Some properties

- **The product rule**: (Gilio & Sanfilippo, 2014)

\[
\mathbb{P}[(B|K) \land (A|H)] = \mathbb{P}[(B|K)|(A|H)] \cdot P(A|H).
\]

Moreover, assuming \(x = P(A|H) > 0\), one has:

\[
\mathbb{P}[(B|K)|(A|H)] = \mu = \frac{\mathbb{P}[(B|K)\land(A|H)]}{P(A|H)} = \frac{z}{x}.
\]

- **Decomposition formula**

\[
B|K = (A|H) \land (B|K) + (\overline{A}|H) \land (B|K).
\]

By the linearity of prevision, and by the product rule, we obtain

\[
P(B|K) = \mathbb{P}[(B|K)|(A|H)]P(A|H) + \mathbb{P}[(B|K)|(\overline{A}|H)]P(\overline{A}|H).
\]
A particular case

If $K = \Omega$ (by replacing $B$ with $C$), then we obtain from the decomposition formula:

$$C = (A|H) \land C + (\bar{A}|H) \land C ,$$

$$P(C) = \mathbb{P}[ C|(A|H) ] P(A|H) + \mathbb{P}[ C|\bar{A}|H ] P(\bar{A}|H)$$

By applying Definition 2, with $K = \Omega$ and by replacing $B$ with $C$, we obtain $C|(A|H) = C \land (A|H) + \mu \bar{A}|H$. That is,

$$C|(A|H) = \begin{cases} 1, & \text{if } AHC \text{ true}, \\ 0, & \text{if } AH\bar{C} \text{ true}, \\ \mu, & \text{if } \bar{A}H \text{ true}, \\ x + \mu(1 - x), & \text{if } HC \text{ true}, \\ \mu(1 - x), & \text{if } H\bar{C} \text{ true}, \end{cases}$$

Notice that $C|(A|H) \neq C|AH$. 
Coherent sets and coherent extensions

- **Theorem (2)**
  
  Let three logically independent events $A, C, H$ be given, with $A \neq \perp$, $H \neq \perp$. The set of all coherent assessments $\mathcal{M} = (x, y, z)$ on $\mathcal{F} = \{A|H, C|(A|H), C|(\overline{A}|H)\}$ is the unit cube $[0, 1]^3$.

  That is, assuming logical independence, every point $(x, y, z) \in [0, 1]^3$ is a coherent assessment on $\{A|H, C|(A|H), C|(\overline{A}|H)\}$.

- **Theorem (3)**

  Given any coherent assessment $(x, y)$ on $\{A|H, C|(A|H)\}$, with $A, C, H$ logically independent, but $A \neq \perp$ and $H \neq \perp$, the extension $z = P(C)$ is coherent if and only if $z \in [z', z'']$, where

  \[ z' = xy \quad \text{and} \quad z'' = xy + 1 - x \quad (7) \]

  That is, we generalized the probabilistic *modus ponens* to the case where the first premise $A$ is replaced by the conditional event $A|H$. 
Proof

From $P(A|H) = x$ and $\mathbb{P}[C|(A|H)] = y$ infer $xy \leq P(C) \leq xy + 1 - x$

From (6), by the linearity of prevision, and the product rule, we obtain

$$z = P(C) = P(A|H) \mathbb{P}[C|(A|H)] + P(\overline{A}|H) \mathbb{P}[C|(\overline{A}|H)]$$

$$= x \cdot y + (1-x) \cdot t$$

From Theorem 3, given any coherent assessment $(x, y)$ on \{A|H, C|(A|H)\}, the extension $t = \mathbb{P}[C|\overline{A}|H]$ on $C|(\overline{A}|H)$ is coherent for every $t \in [0,1]$. As $z = xy + (1-x) t$, it follows that

$$xy \leq P(C) \leq xy + (1-x)$$

if $t=0$ \hspace{1cm} if $t=1$

Prevision entailment:

$$P(A|H) = 1, \ \& \ \mathbb{P}[C|(A|H)] = 1 \implies P(C) = 1$$
Conclusion

- We generalized the probabilistic modus ponens in terms of conditional random quantities in the setting of coherence.
- Specifically, we replaced the categorical premise $A$ and the antecedent $A$ of the conditional premise $C|A$ by the conditional event $A|H$.
- We proved a generalized decomposition formula for conditional events and we gave some results.
- We propagated the previsions from the premises of the generalized probabilistic modus ponens to the conclusion.
- We have shown that the lower and the upper bounds on the conclusion of the generalized probabilistic modus ponens coincide with the respective bounds on the conclusion for the (non-nested) probabilistic modus ponens.
Further work

- We will study other instantiations to obtain further generalizations of modus ponens, e.g., by also replacing the consequent $C$ of the conditional premise $C|A$ and the conclusion $C$ by a conditional event $C|K$: from $\{A|H, (C|K)|(A|H)\}$ infer $C|K$.

- We will focus on similar generalizations (also involving imprecision) of other argument forms like the probabilistic modus tollens.

- We will study similar generalization of inference rules in System P where a conditional will be replaced by an iterated one.
Thank you for your attention!

Link to the full paper

I would also like to thank the organizers of ECSQARU 2017, and in particular Alessandro Antonucci, for giving me this opportunity to talk by skype.
Some references


