Concepts for decision making under severe uncertainty with partial ordinal and partial cardinal preferences

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Motivation

- Stochastic dominance (first order)
- imprecise decision criteria
- Hurwicz
- Maximin

Comparability of exchanges
Comparability of alternatives
Probabilistic information
Full
No

SEUT
A patient $P_1$ has symptoms possibly caused by one of the chronic diseases $D_1$ or $D_2$. There are two different types of medication, $M_1$ and $M_2$, available.

For another patient $P_2$ suffering from the same symptoms there are instead medications $M_1^*$ and $M_2^*$ available.

The situations are described in the following tables:

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>death</td>
<td>cure</td>
</tr>
<tr>
<td>$M_2$</td>
<td>abatement 30%</td>
<td>ab. 20%</td>
</tr>
</tbody>
</table>

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<tr>
<th></th>
<th>$D_1$</th>
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<tbody>
<tr>
<td>$M_1^*$</td>
<td>ab. 10%</td>
<td>cure</td>
</tr>
<tr>
<td>$M_2^*$</td>
<td>ab. 30%</td>
<td>ab. 20%</td>
</tr>
</tbody>
</table>

Approaching the situation intuitively:

Left: Choosing "Maximin-medication" $M_2$ seems to be reasonable.

Right: Choosing "Maximin-medication" $M_2^*$ might seem counter-intuitive (or at least less obvious).
In order to formally capture the difference in the two situations discussed in the beginning, we start by defining the following concept:

**Definition: Preference System**

Let $A$ be a non-empty set and let $R_1 \subset A \times A$ denote a preorder (i.e. reflexive and transitive) on $A$. Moreover, let $R_2 \subset R_1 \times R_1$ denote a preorder on $R_1$. Then the triplet $A = [A, R_1, R_2]$ is called a preference system on $A$.

**Interpretation:** For elements $a, b, c, d \in A$

- $(a, b) \in R_1$ means alternative $a$ is weakly preferred to alternative $b$.
- $((a, b), (c, d)) \in R_2$ means that exchanging alternative $b$ by alternative $a$ is weakly preferred to exchanging alternative $d$ by alternative $c$. 
Example, continued

Patient 1

\[ M_1 \]

\[ M_2 \]

\[ \text{abatement 30\%} \]

\[ \text{abatement 20\%} \]

\[ \text{death} \]

Patient 2

\[ M_1^* \]

\[ M_2^* \]

\[ \text{abatement 30\%} \]

\[ \text{abatement 20\%} \]

\[ \text{ab. 10\%} \]
Example, continued

Metarelation $R_2$ contains edge $(cure, 20\%) P_{R_2} (30\%, 10\%)$
For what follows, we restrict our analysis on preference systems satisfying a certain property of consistency (implying compatibility of $R_1$ and $R_2$). Precisely, we have

**Definition: Consistency**

A preference system $\mathcal{A}$ is **consistent** if there exists a function $u : A \rightarrow [0, 1]$ such that for all $a, b, c, d \in A$ the following two properties hold:

i) If $(a, b) \in R_1$, then $u(a) \geq u(b)$ with equality iff $(a, b) \in I_{R_1}$.

ii) If $((a, b), (c, d)) \in R_2$, then $u(a) - u(b) \geq u(c) - u(d)$ with equality iff $((a, b), (c, d)) \in I_{R_2}$.

Every such function $u$ is then said to **(weakly) represent** the preference system $\mathcal{A}$. The set of all (weak) representations $u$ of $\mathcal{A}$ is denoted by $\mathcal{U}_A$. 
Checking consistency via linear programming

Proposition: Checking consistency

Let \( \mathcal{A} = [A, R_1, R_2] \) be a preference system, where \( A = \{a_1, \ldots, a_n\} \) is a finite and non-empty set. Consider the linear optimization problem

\[
\begin{align*}
\varepsilon &= \langle (0, \ldots, 0, 1)', (u_1, \ldots, u_n, \varepsilon)' \rangle \rightarrow \max_{(u_1, \ldots, u_n, \varepsilon) \in \mathbb{R}^{n+1}} (u_1, \ldots, u_n, \varepsilon) \\
\end{align*}
\]

(1)

with constraints \( 0 \leq (u_1, \ldots, u_n, \varepsilon) \leq 1 \) and

i) \( u_p = u_q \) for all \( (a_p, a_q) \in I_{R_1} \setminus \text{diag}(A) \)

ii) \( u_q + \varepsilon \leq u_p \) for all \( (a_p, a_q) \in P_{R_1} \)

iii) \( u_p - u_q = u_r - u_s \) for all \( ((a_p, a_q), (a_r, a_s)) \in I_{R_2} \setminus \text{diag}(R_1) \)

iv) \( u_r - u_s + \varepsilon \leq u_p - u_q \) for all \( ((a_p, a_q), (a_r, a_s)) \in P_{R_2} \)

Then \( \mathcal{A} \) is consistent if and only if the optimal outcome of (1) is strictly positive.
We now turn to decision theory under complex uncertainty with acts taking values in a preference system (ps). First, we need some additional notation:

- \((S, \sigma(S))\): set of states equipped with suitable \(\sigma\)-field
- \(\mathcal{M}\): credal set of all probability measures on \((S, \sigma(S))\) compatible with the available (partial) probabilistic information

For a given consistent preference system \(\mathcal{A}\), we call every mapping \(X : S \rightarrow A\) a ps-valued act. Moreover, we define \(\mathcal{F}(\mathcal{A}, \mathcal{M}, S) \subset A^S := \{f | f : S \rightarrow A\}\) by setting

\[
\mathcal{F}(\mathcal{A}, \mathcal{M}, S) := \left\{ X \in A^S : u \circ X \text{ is } \sigma(S)\-\mathcal{B}_\mathbb{R}\text{-measurable for all } u \in \mathcal{U}_A \right\}
\]

Given this notation, we can now define our main object of study:

**Definition: Decision System**

A subset \(\mathcal{G} \subset \mathcal{F}(\mathcal{A}, \mathcal{M}, S)\) is called **decision system** (with information base \((\mathcal{A}, \mathcal{M})\)). Moreover, call \(\mathcal{G}\) finite, if both \(|\mathcal{G}| < \infty\) and \(|S| < \infty\).
Consider again the scenario for patient 2:

\[
\begin{array}{c|c|c}
D_1 & D_2 \\
\hline
M_1^* & a_1 := \text{ab. 10\%} & a_2 := \text{cure} \\
M_2^* & a_3 := \text{ab. 30\%} & a_4 := \text{ab. 20\%} \\
\end{array}
\]

Moreover, suppose we have the information that disease $D_2$ is more likely than disease $D_1$, i.e. probabilistic information is described by the credal set

\[
\mathcal{M} = \left\{ \pi \in \mathcal{P} \left( \{D_1, D_2\} \right) \mid \pi(\{D_1\}) \leq \pi(\{D_2\}) \right\}
\]

Finally, the preference system $\mathcal{A} = \{\{a_1, a_2, a_3, a_4\}, R_1, R_2\}$ where

- $R_1$ induced by $a_2 P_{R_1} a_3 P_{R_1} a_4 P_{R_1} a_1$
- $P_{R_2} = \{((a_2, a_4), (a_3, a_1))\}$ consists of one single edge

Then $\mathcal{G} = \{M_1^*, M_2^*\}$ defines a decision system with information base $(\mathcal{A}, \mathcal{M})$. 
How to utilize the information base?

Given a decision system $G$, our goal is to choose a subset $G_{opt} \subset G$ of 'optimal' acts in a way best possibly utilizing the available information specified by $(\mathcal{A}, \mathcal{M})$.

In the following, we discuss three different approaches:

- **Numerical representations**: Assign a real number, based on a generalized expected value, to each act and choose those acts with the highest values.

- **Global comparisons**: E.g., choose an act $X$ if there exists (global) $(u, \pi)$ compatible with $(\mathcal{A}, \mathcal{M})$ with respect to which $X$ dominates all other acts in expectation.

- **Pairwise comparisons**: E.g., choose an act $X$ if, for all other acts $Y$, there exists $(u_Y, \pi_Y)$ compatible with $(\mathcal{A}, \mathcal{M})$ with respect to $X$ dominates $Y$ in expectation.
Approach 2: Criteria based on Global Comparisons

We now turn to the first of two approaches not needing the specification of the granularity parameter $\delta$.

**Approach 2: Decision criteria**

Let $G \subseteq F_{(A,M,S)}$ denote a decision system. We call an act $X \in G$

i) $A|M$-admissible : iff

$$\exists u \in U_A \ \exists \pi \in M \ \forall Y \in G : \ E_{\pi}(u \circ X) \geq E_{\pi}(u \circ Y)$$

ii) $A$-admissible : iff

$$\exists u \in U_A \ \forall \pi \in M \ \forall Y \in G : \ E_{\pi}(u \circ X) \geq E_{\pi}(u \circ Y)$$

iii) $M$-admissible : iff

$$\exists \pi \in M \ \forall u \in U_A \ \forall Y \in G : \ E_{\pi}(u \circ X) \geq E_{\pi}(u \circ Y)$$

iv) $A|M$-dominant : iff

$$\forall u \in U_A \ \forall \pi \in M \ \forall Y \in G : \ E_{\pi}(u \circ X) \geq E_{\pi}(u \circ Y)$$
Example, continued

$\mathbb{E}_\pi(M_1^*) = \pi_1 \cdot (\rightarrow) + \pi_2 \cdot (\rightarrow + \rightarrow)$

$\geq 0$ for $\pi_2 \geq \pi_1$

$\geq \pi_2 \cdot (\rightarrow)$

$= \pi_2 \cdot (\rightarrow) + \pi_1 \cdot 0 = \mathbb{E}_\pi(M_2^*)$

$\Rightarrow$ medication $M_2^*$ is not $A|M$-admissible!

Metarelation $R_2$ contains edge
(cure, 20%) $P_{R_2}$ (30%, 10%)

$\rightarrow + \rightarrow > \rightarrow + \rightarrow$
Approach 3: Criteria based on Pairwise Comparisons

Finally, we consider a local approach. We define six binary relations $R_{\exists\exists}, R_{\exists\forall}^1, R_{\exists\forall}^2, R_{\forall\exists}^1, R_{\forall\exists}^2$ and $R_{\forall\forall}$ on $\mathcal{F}(A,\mathcal{M},S)$ by setting for all $X, Y \in \mathcal{F}(A,\mathcal{M},S)$:

$$
(X, Y) \in R_{\exists\exists} : \iff \exists u \in \mathcal{U}_A \ \exists \pi \in \mathcal{M} : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

$$
(X, Y) \in R_{\exists\forall}^1 : \iff \exists u \in \mathcal{U}_A \ \forall \pi \in \mathcal{M} : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

$$
(X, Y) \in R_{\exists\forall}^2 : \iff \exists \pi \in \mathcal{M} \ \forall u \in \mathcal{U}_A : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

$$
(X, Y) \in R_{\forall\exists}^1 : \iff \forall u \in \mathcal{U}_A \ \exists \pi \in \mathcal{M} : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

$$
(X, Y) \in R_{\forall\exists}^2 : \iff \forall \pi \in \mathcal{M} \ \exists u \in \mathcal{U}_A : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

$$
(X, Y) \in R_{\forall\forall} : \iff \forall \pi \in \mathcal{M} \ \forall u \in \mathcal{U}_A : E_\pi(u \circ X) \geq E_\pi(u \circ Y)
$$

Definition: Local admissibility

Let $R \in \{R_{\exists\exists}, R_{\exists\forall}^1, R_{\exists\forall}^2, R_{\forall\exists}^1, R_{\forall\exists}^2, R_{\forall\forall}\} =: \mathcal{R}_p$. We call $X \in \mathcal{G}$ **locally admissible** w.r.t. $R$, if it is an element of $\max_R(\mathcal{G}) := \{Y \in \mathcal{G} : \nexists Z \in \mathcal{G} \text{ s.t. } (Z, Y) \in P_R\}$. 
We now discuss some special cases of the relations just defined:

\[ \mathcal{U}_A \] is a class of plts: The \( R \)-locally admissible acts w.r.t. relations \( R \in \mathcal{R}_p \) containing

- \( \exists \pi \in \mathcal{M} \ldots \) coincide with the acts that are optimal in the sense of Walley's maximality.
- \( \forall \pi \in \mathcal{M} \ldots \) coincide with the acts that are optimal in the sense of Bewley's dominance.
Approach 3: Some special cases

We now discuss some special cases of the relations just defined:

\[ \mathcal{M} = \{ \pi \} \text{ is a singleton: The relations containing } \ldots \forall u \in \mathcal{U}_A \ldots \text{ reduce to} \]

- first order stochastic dominance if \( R_2 = \emptyset \).
- SEUT if \( R_1 \) and \( R_2 \) are complete and 'compatible'.
- second order SD if \( R_1 \) is complete and \( R_2 \) appropriately models decreasing returns to scale.
Summary

- Introduced preference systems as tools for modeling partially ordinal and partially cardinal preference structures.

- Proposed three approaches for decision making with ps-valued acts:
  1. Numerical representations based on generalized expectation intervals
  2. Criteria induced by pairwise comparisons of acts
  3. Criteria induced by global (simultaneous) comparisons of acts

- Provided linear programming based algorithms for checking optimality of acts with respect to the proposed criteria.