Concepts for decision making under severe uncertainty with partial ordinal and partial cardinal preferences

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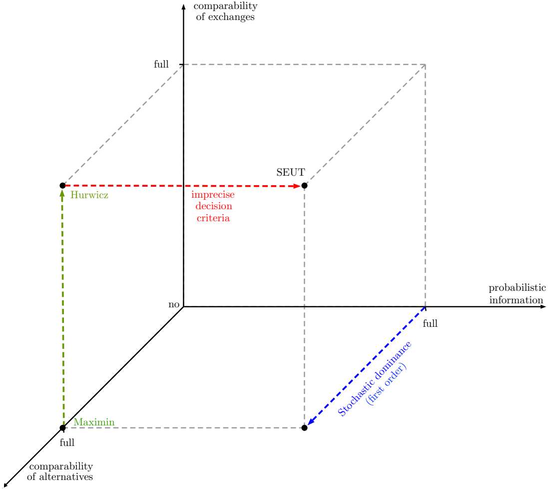
Motivation


comparability
of exchanges
full
SEUT
Hurwicz
imprecise
decision
criteria
no
probabilistic
information
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Stochastic dominance
(first order)
Maximin
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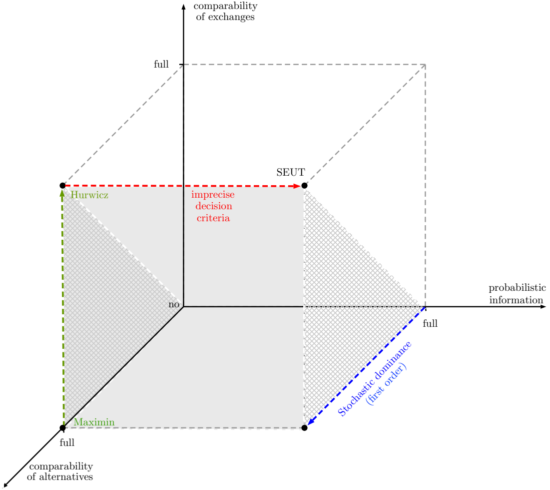
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Example: Choosing the right medication

- A patient P_1 has symptoms possibly caused by one of the chronic diseases D_1 or D_2. There are two different types of medication, M_1 and M_2, available.
- For another patient P_2 suffering from the same symptoms there are instead medications M_1^* and M_2^* available.

The situations are described in the following tables:

	D_1	D_2
M_1	death	cure
M_2	abatement 30%	ab. 20%

	D_1	D_2
M_1^*	ab. 10%	cure
M_2^*	ab. 30%	ab. 20%

Approaching the situation intuitively:

- Left: Choosing "Maximin-medication" M_2 seems to be reasonable.
- Right: Choosing "Maximin-medication" M_2^* might seem counter-intuitive (or at least less obvious).

Setting the stage

In order to formally capture the difference in the two situations discussed in the beginning, we start by defining the following concept:

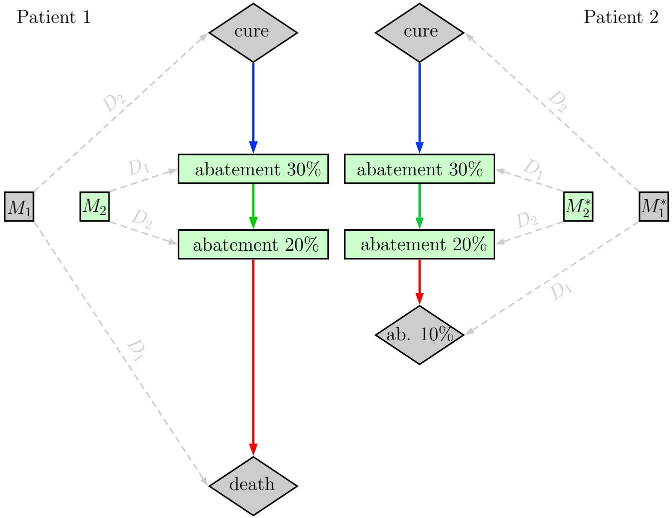
Definition: Preference System

Let A be a non-empty set and let $R_1 \subset A \times A$ denote a preorder (i.e. reflexive and transitive) on A. Moreover, let $R_2 \subset R_1 \times R_1$ denote a preorder on R_1. Then the triplet $\mathcal{A} = [A, R_1, R_2]$ is called a **preference system** on A.

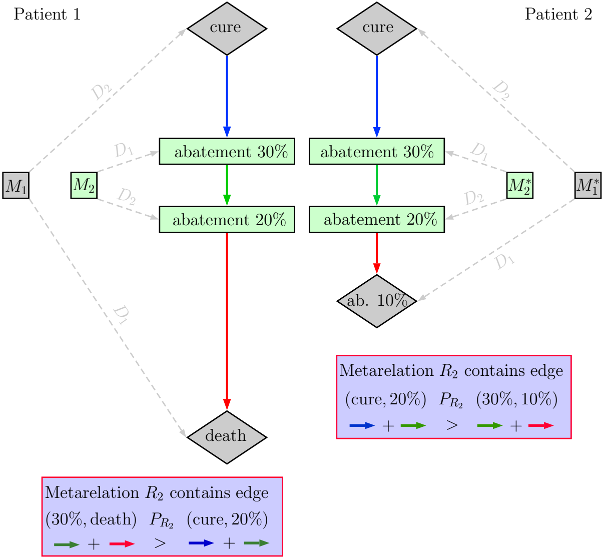
Interpretation: For elements $a, b, c, d \in A$

- $(a, b) \in R_1$ means alternative a is weakly preferred to alternative b.
- $((a, b), (c, d)) \in R_2$ means that exchanging alternative b by alternative a is weakly preferred to exchanging alternative d by alternative c.

Example, continued



Example, continued



Setting the stage, continued

For what follows, we restrict our analysis on preference systems satisfying a certain property of consistency (implying compatibility of R_1 and R_2). Precisely, we have

Definition: Consistency

A preference system $\mathcal{A}$ is **consistent** if there exists a function $u : A \to [0,1]$ such that for all $a, b, c, d \in A$ the following two properties hold:

i) If $(a, b) \in R_1$, then $u(a) \geq u(b)$ with equality iff $(a, b) \in I_{R_1}$.

ii) If $((a, b), (c, d)) \in R_2$, then $u(a) - u(b) \geq u(c) - u(d)$ with equality iff $((a, b), (c, d)) \in I_{R_2}$.

Every such function u is then said to **(weakly) represent** the preference system $\mathcal{A}$. The set of all (weak) representations u of $\mathcal{A}$ is denoted by $\mathcal{U}_{\mathcal{A}}$.

Checking consistency via linear programming

Proposition: Checking consistency

Let $\mathcal{A} = [A, R_1, R_2]$ be a preference system, where $A = \{a_1, \ldots, a_n\}$ is a finite and non-empty set. Consider the linear optimization problem

$$\varepsilon = \langle (0, \ldots, 0, 1)', (u_1, \ldots, u_n, \varepsilon)' \rangle \longrightarrow \max_{(u_1, \ldots, u_n, \varepsilon) \in \mathbb{R}^{n+1}} \tag{1}$$

with constraints $0 \leq (u_1, \ldots, u_n, \varepsilon) \leq 1$ and

i) $u_p = u_q$ for all $(a_p, a_q) \in I_{R_1} \setminus \mathrm{diag}(A)$

ii) $u_q + \varepsilon \leq u_p$ for all $(a_p, a_q) \in P_{R_1}$

iii) $u_p - u_q = u_r - u_s$ for all $((a_p, a_q), (a_r, a_s)) \in I_{R_2} \setminus \mathrm{diag}(R_1)$

iv) $u_r - u_s + \varepsilon \leq u_p - u_q$ for all $((a_p, a_q), (a_r, a_s)) \in P_{R_2}$

Then $\mathcal{A}$ is consistent if and only if the optimal outcome of (1) is strictly positive.

Decision making with ps-valued acts: Basic setting

We now turn to decision theory under complex uncertainty with acts taking values in a preference system (ps). First, we need some additional notation:

- $(S, \sigma(S))$: set of states equipped with suitable σ-field
- $\mathcal{M}$: credal set of all probability measures on $(S, \sigma(S))$ compatible with the available (partial) probabilistic information

For a given consistent preference system $\mathcal{A}$, we call every mapping $X : S \to A$ a ps-valued act. Moreover, we define $\mathcal{F}_{(\mathcal{A},\mathcal{M},S)} \subset A^S := \{f | f : S \to A\}$ by setting

$$\mathcal{F}_{(\mathcal{A},\mathcal{M},S)} := \Big\{ X \in A^S : u \circ X \text{ is } \sigma(S)\text{-}\mathcal{B}_{\mathbb{R}}\text{-measurable for all } u \in \mathcal{U}_{\mathcal{A}} \Big\}$$

Given this notation, we can now define our main object of study:

Definition: Decision System

A subset $\mathcal{G} \subset \mathcal{F}_{(\mathcal{A},\mathcal{M},S)}$ is called **decision system** (with information base $(\mathcal{A}, \mathcal{M})$). Moreover, call $\mathcal{G}$ finite, if both $|\mathcal{G}| < \infty$ and $|S| < \infty$.

Example, continued

Consider again the scenario for patient 2:

	D_1	D_2
M_1^*	$a_1 :=$ ab. 10%	$a_2 :=$ cure
M_2^*	$a_3 :=$ ab. 30%	$a_4 :=$ ab. 20%

Moreover, suppose we have the information that disease D_2 is more likely than disease D_1, i.e. probabilistic information is described by the credal set

$$\mathcal{M} = \Big\{\pi \in \mathcal{P}(\{D_1, D_2\}) \mid \pi(\{D_1\}) \leq \pi(\{D_2\})\Big\}$$

Finally, the preference system $\mathcal{A} = [\{a_1, a_2, a_3, a_4\}, R_1, R_2]$ where

- R_1 induced by $a_2 P_{R_1} a_3 P_{R_1} a_4 P_{R_1} a_1$
- $P_{R_2} = \{((a_2, a_4), (a_3, a_1))\}$ consists of one single edge

Then $\mathcal{G} = \{M_1^*, M_2^*\}$ defines a decision system with information base $(\mathcal{A}, \mathcal{M})$.

How to utilize the information base?

Given a decision system $\mathcal{G}$, our goal is to choose a subset $\mathcal{G}_{opt} \subset \mathcal{G}$ of 'optimal' acts in a way best possibly utilizing the available information specified by $(\mathcal{A}, \mathcal{M})$.

In the following, we discuss three different approaches:

- Numerical representations: Assign a real number, based on a generalized expected value, to each act and choose those acts with the highest values.
- Global comparisons: E.g., choose an act X if there exists (global) (u, π) compatible with $(\mathcal{A}, \mathcal{M})$ with respect to which X dominates *all* other acts in expectation.
- Pairwise comparisons: E.g., choose an act X if, for all other acts Y, there exists (u_Y, π_Y) compatible with $(\mathcal{A}, \mathcal{M})$ with respect to X dominates Y in expectation.

Approach 2: Criteria based on Global Comparisons

We now turn to the first of two approaches not needing the specification of the granularity parameter δ.

Approach 2: Decision criteria

Let $\mathcal{G} \subset \mathcal{F}_{(\mathcal{A},\mathcal{M},S)}$ denote a decision system. We call an act $X \in \mathcal{G}$

i) $\mathcal{A}|\mathcal{M}-$**admissible** :iff

$$\exists u \in \mathcal{U}_{\mathcal{A}} \; \exists \pi \in \mathcal{M} \; \forall Y \in \mathcal{G} : \; \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y)$$

ii) $\mathcal{A}-$**admissible** :iff

$$\exists u \in \mathcal{U}_{\mathcal{A}} \; \forall \pi \in \mathcal{M} \; \forall Y \in \mathcal{G} : \; \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y)$$

iii) $\mathcal{M}-$**admissible** :iff

$$\exists \pi \in \mathcal{M} \; \forall u \in \mathcal{U}_{\mathcal{A}} \; \forall Y \in \mathcal{G} : \; \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y)$$

iv) $\mathcal{A}|\mathcal{M}-$**dominant** :iff

$$\forall u \in \mathcal{U}_{\mathcal{A}} \; \forall \pi \in \mathcal{M} \; \forall Y \in \mathcal{G} : \; \mathbb{E}_{\pi}(u \circ X) \geq \mathbb{E}_{\pi}(u \circ Y)$$

$$\mathbb{E}_\pi(M_1^*) = \pi_1 \cdot (-\textcolor{red}{\longrightarrow}) + \pi_2 \cdot (\textcolor{green}{\longrightarrow} + \textcolor{blue}{\longrightarrow})$$

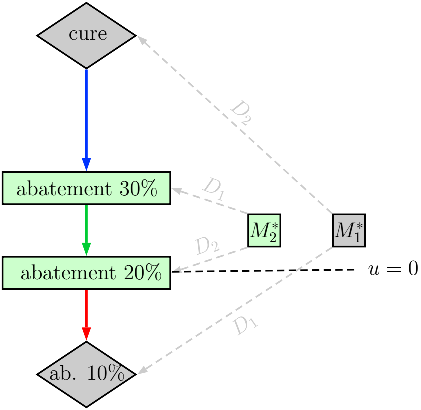
$$> \pi_1 \cdot (-\textcolor{red}{\longrightarrow}) + \pi_2 \cdot (\textcolor{green}{\longrightarrow} + \textcolor{red}{\longrightarrow})$$

$$= \underbrace{\textcolor{red}{\longrightarrow} \cdot (\pi_2 - \pi_1)}_{\geq 0 \text{ for } \pi_2 \geq \pi_1} + \pi_2 \cdot (\textcolor{green}{\longrightarrow})$$

$$\geq \pi_2 \cdot (\textcolor{green}{\longrightarrow})$$

$$= \pi_2 \cdot (\textcolor{green}{\longrightarrow}) + \pi_1 \cdot 0 = \mathbb{E}_\pi(M_2^*)$$

$\Rightarrow$ medication M_2^* is not $\mathcal{A}|\mathcal{M}$-admissible!



Metarelation R_2 contains edge

$(\text{cure}, 20\%) \quad P_{R_2} \quad (30\%, 10\%)$

$\textcolor{blue}{\longrightarrow} + \textcolor{green}{\longrightarrow} \quad > \quad \textcolor{green}{\longrightarrow} + \textcolor{red}{\longrightarrow}$

Approach 3: Criteria based on Pairwise Comparisons

Finally, we consider a local approach. We define six binary relations $R_{\exists\exists}, R^1_{\exists\forall}, R^2_{\exists\forall}, R^1_{\forall\exists}, R^2_{\forall\exists}$ and $R_{\forall\forall}$ on $\mathcal{F}_{(\mathcal{A},\mathcal{M},\mathcal{S})}$ by setting for all $X, Y \in \mathcal{F}_{(\mathcal{A},\mathcal{M},\mathcal{S})}$:

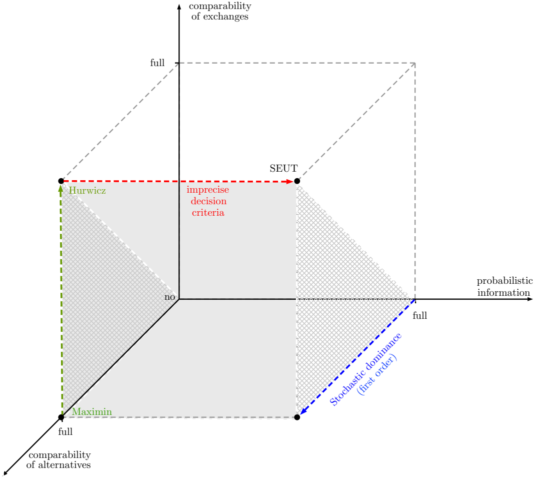
$$\begin{aligned}
(X,Y) \in R_{\exists\exists} \quad &:\Leftrightarrow \quad \exists u \in \mathcal{U}_{\mathcal{A}}\ \exists \pi \in \mathcal{M} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y) \\
(X,Y) \in R^1_{\exists\forall} \quad &:\Leftrightarrow \quad \exists u \in \mathcal{U}_{\mathcal{A}}\ \forall \pi \in \mathcal{M} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y) \\
(X,Y) \in R^2_{\exists\forall} \quad &:\Leftrightarrow \quad \exists \pi \in \mathcal{M}\ \forall u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y) \\
(X,Y) \in R^1_{\forall\exists} \quad &:\Leftrightarrow \quad \forall u \in \mathcal{U}_{\mathcal{A}}\ \exists \pi \in \mathcal{M} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y) \\
(X,Y) \in R^2_{\forall\exists} \quad &:\Leftrightarrow \quad \forall \pi \in \mathcal{M}\ \exists u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y) \\
(X,Y) \in R_{\forall\forall} \quad &:\Leftrightarrow \quad \forall \pi \in \mathcal{M}\ \forall u \in \mathcal{U}_{\mathcal{A}} : \mathbb{E}_\pi(u \circ X) \geq \mathbb{E}_\pi(u \circ Y)
\end{aligned}$$

Definition: Local admissibility

Let $R \in \{R_{\exists\exists}, R^1_{\exists\forall}, R^2_{\exists\forall}, R^1_{\forall\exists}, R^2_{\forall\exists}, R_{\forall\forall}\} =: \mathcal{R}_p$. We call $X \in \mathcal{G}$ **locally admissible** w.r.t. R, if it is an element of $\max_R(\mathcal{G}) := \{Y \in \mathcal{G} : \nexists Z \in \mathcal{G} \text{ s.t. } (Z,Y) \in P_R\}$.

Approach 3: Some special cases

We now discuss some special cases of the relations just defined:



$\mathcal{U}_{\mathcal{A}}$ is a class of plts: The R-locally admissible acts w.r.t. relations $R \in \mathcal{R}_p$ containing

- $\ldots \exists \pi \in \mathcal{M} \ldots$ coincide with the acts that are optimal in the sense of Walley's maximality.
- $\ldots \forall \pi \in \mathcal{M} \ldots$ coincide with the acts that are optimal in the sense of Bewley's dominance.

Approach 3: Some special cases

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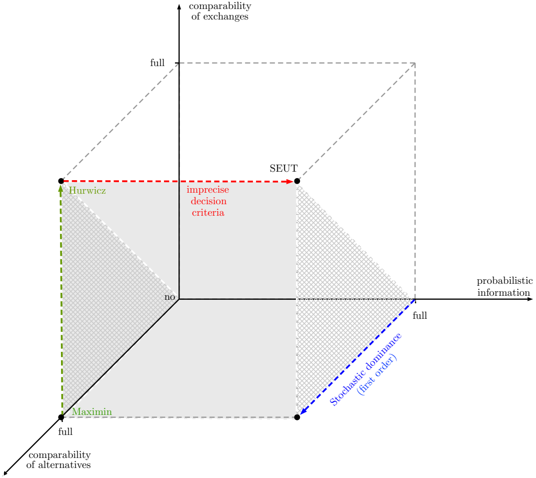

$\mathcal{M} = \{\pi\}$ is a singleton: The relations containing $\ldots \forall u \in \mathcal{U}_{\mathcal{A}} \ldots$ reduce to

- first order stochastic dominance if $R_2 = \emptyset$.
- SEUT if R_1 and R_2 are complete and 'compatible'.
- second order SD if R_1 is complete and R_2 appropriately models decreasing returns to scale.

Summary

- Introduced preference systems as tools for modeling partially ordinal and partially cardinal preference structures
- Proposed three approaches for decision making with ps-valued acts:
  i) Numerical representations based on generalized expectation intervals
  ii) Criteria induced by pairwise comparisons of acts
  iii) Criteria induced by global (simultaneous) comparisons of acts
- provided linear programming based algorithms for checking optimality of acts with respect to the proposed criteria