Implementation of linear core-based criterion for testing extreme exact games

Václav Kratochvíl, Milan Studený

Institute of Information Theory and Automation of the CAS, Prague

velorex@utia.cas.cz, studeny@utia.cas.cz

Notation

Let N be a finite non-empty set of *variables*, $|N| \geq 2$, $\mathcal{P}(N) := \{S : S \subseteq N\}$. $\mathbb{R}^N$ will denote the set of real vectors whose components are indexed by elements of N. A set function m is l-standardized if $\forall |S| \leq 1 : m(S) = 0$.

Exact games & lower probabilities

Let $m : \mathcal{P}(N) \to \mathbb{R}$, $m(\emptyset) = 0$ be a game. Its *core* is a polytope $C(m)$ in $\mathbb{R}^N$ defined by

$$\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = m(N) \text{ \& } \forall S \subseteq N \sum_{i \in S} x_i \geq m(S)\}.$$

If $C(m) \neq \emptyset$ then we say the game is exact if

$$\forall S \subseteq N \; \exists x \in C(m) \; \sum_{i \in S} x_i = m(S).$$

Non-negative exact game normalized by $m(N) = 1$ is a coherent lower probability.

Polyhedral cone

The collection of exact games is a rational polyhedral cone. Moreover, l-standardized exact games form a pointed rational cone $E_l(N)$. Every pointed polyhedral cone has finitely many extreme rays.

Extreme exact game

An l-standardized exact game is called *extreme* if it generates an extreme ray of $E_l(N)$. Since $E_l(N)$ is a rational cone, every extreme game is a multiple of an integer-valued game $m : \mathcal{P}(N) \to \mathbb{R}$. We can limit ourselves to **integer-valued games**. This is important for the implementation and we offer a criterion testable on a computer.

Feasible min-representations

A min-representation of a game m is a finite set $\mathcal{R} \subseteq \mathbb{R}^N$ such that $m(S) = \min_{x \in \mathcal{R}} \sum_{i \in S} x_i$ for any $S \subseteq N$.
A game is exact $\Leftrightarrow$ it has a feasible min-representation, that is, $\sum_{i \in N} x_i = m(N)$ holds for any $x \in \mathcal{R}$.

Standard min-representation

An exceptional **standard min-representation** $\overline{\mathcal{R}}$ of an exact game m consists of (the set of) vertices of the core $C(m)$.

Tightness structure

Let $\mathcal{R} \subseteq \mathbb{R}^N$ be a feasible min-representation of an exact game. For every vector $x \in \mathcal{R}$ a collection of sets

$$\mathcal{T}_x^m = \{S \subseteq N : \sum_{i \in S} x_i = m(S)\}$$

is called its **tightness class.** List of all tightness classes for all vectors from $\mathcal{R}$ is called **tightness structure** $\mathcal{T}^{\mathcal{R}}$.
Given two feasible min-representations $\mathcal{R}, \mathcal{L}$ of m we say that the $\mathcal{T}^{\mathcal{R}}$ **refines** $\mathcal{T}^{\mathcal{L}}$ if

$$\forall x \in \mathcal{R} \; \exists y \in \mathcal{L} : \; \mathcal{T}_x^{\mathcal{R}} \subseteq \mathcal{T}_y^{\mathcal{L}}.$$

The standard min-representation has the coarsest tightness structure.

Finest min-representation

There exist the finest min-representations = representations with the finest tightness structure. The finest min-representations are not unique but the corresponding tightness structure is unique. We have an algorithm to obtain it!

Reduced dimension

For each *finest min-representation* we can compute the so-called **reduced dimension** (technical details omitted).
Let $\mathcal{R}, \mathcal{L}$ be two feasible min-representation such that $\mathcal{T}^{\mathcal{R}}$ refines $\mathcal{T}^{\mathcal{L}}$. Then

$$1 \leq \text{redim}(\mathcal{L}) \leq \text{redim}(\mathcal{R}) \leq 2^{|N|} - |N| - 1.$$

Generally, reduced dimension is the dimension of the smallest face of the pointed polyhedral cone containing respective game.

Criterion

An l-standardized game is extreme if the reduced dimension of its finest min-representation is 1.

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Extreme exact game - A

∅	a	b	c	ab	ac	bc	abc
0	0	0	0	1/2	1/2	1/2	1

∅	a	b	c	ab	ac	bc	abc
0	0	0	0	1	1	1	2

Standard min-representation of the game

a	b	c
1/2	0	1/2
1/2	1/2	0
0	1/2	1/2

a	b	c
1	0	1
1	1	0
0	1	1

Non-extreme exact game - B

∅	a	b	c	ab	ac	bc	abc
0	0	0	0	2/11	3/11	6/11	1

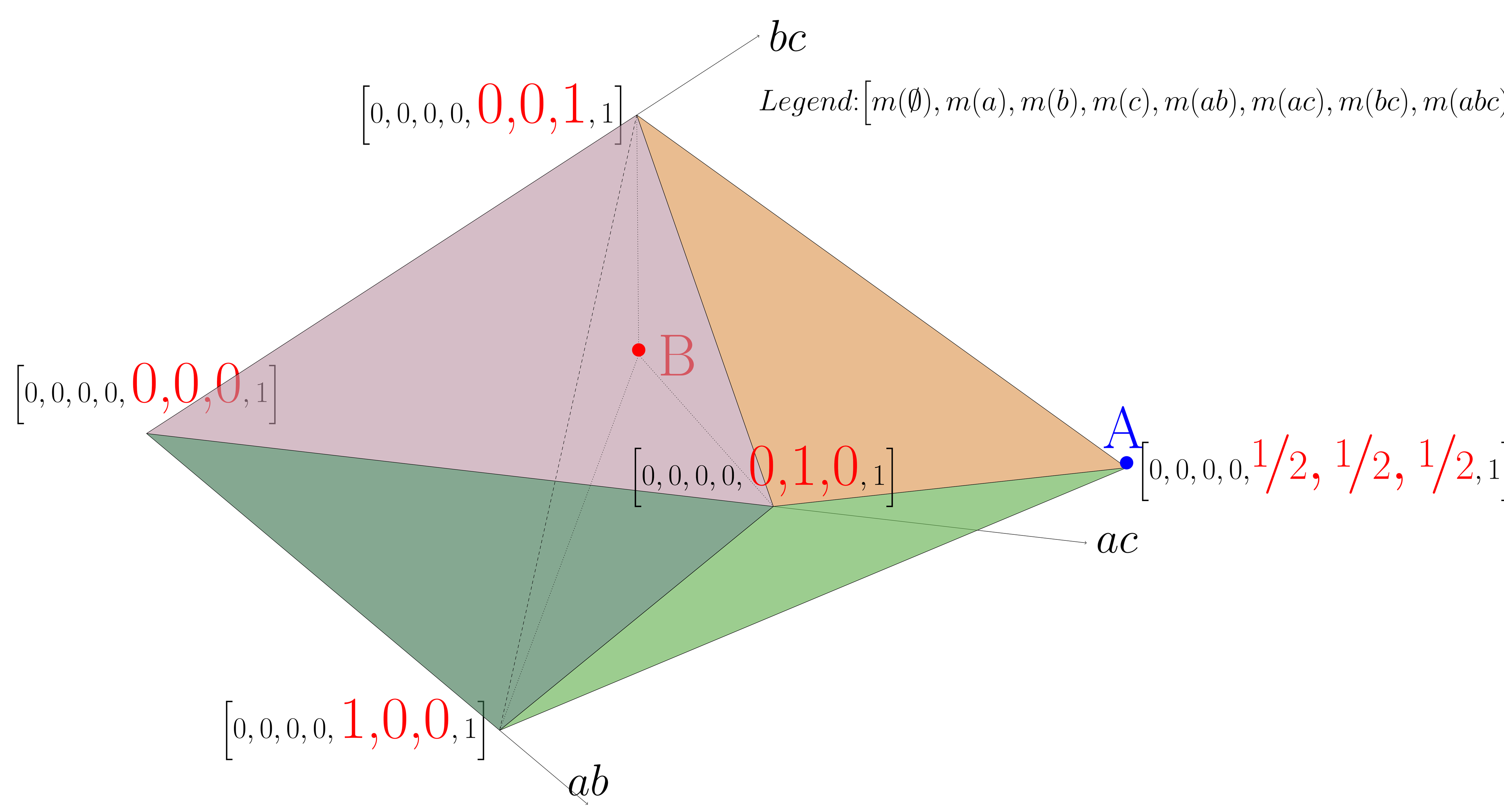
∅	a	b	c	ab	ac	bc	abc
0	0	0	0	2	3	6	11

Standard min-representation of the game

a	b	c
0	2/11	9/11
0	8/11	3/11
5/11	6/11	0
3/11	8/11	0
2/11	0	9/11
5/11	0	6/11

a	b	c
0	2	9
0	8	3
5	6	0
3	8	0
2	0	9
5	0	6

Polyhedral cone - 3D visualization on selected coordinates



$N = 4$ example

Extreme exact game

∅	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd
0	0	0	0	0	2	1	0	1	0	0	2	2	1	1	3

Reduced dimension

Reduced dimension of the finest min-representation equals to 1. The game is extreme.

Tightness structure(s)

$\mathcal{R}$	∅	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	abcd
(2,0,1,0)	•		•		•	•			•	•			•		•	•
(0,2,1,0)	•	•			•	•	•	•					•	•		•
(1,1,0,1)	•			•		•	•		•			•				•
(1,2,0,0)	•			•	•		•				•			•		•
(2,1,0,0)	•			•	•				•		•				•	•
$(\frac{3}{2},\frac{3}{2},0,0)$	•			•	•						•					•

Software - http://gogo.utia.cas.cz/finest-min-representation/