(Irrelevant) natural extension of choice functions

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Rejection functions on gambles

Gambles The random variable X takes values in the finite possibility space $\mathscr{X}$. Any real-valued function on $\mathscr{X}$ is called a **gamble**, and we collect all of them in $\mathscr{L}(\mathscr{X})$ (or $\mathscr{L}$). Given two gambles f and g in $\mathscr{L}$, we say that $f \leq g$ if $(\forall x \in \mathscr{X}) f(x) \leq g(x)$. Its strict variant $<$ on $\mathscr{L}$ is given by: $f < g \Leftrightarrow (f \leq g$ and $f \neq g)$; we collect all f such that $0 < f$ in $\mathscr{L}_{>0}$.
We define $\mathscr{Q} \subseteq \mathscr{P}(\mathscr{L})$ as the collection of **non-empty but finite subsets** of $\mathscr{L}$.
Rejection function A **rejection function** R is a map

$$R\colon \mathscr{Q} \to \mathscr{Q} \cup \{\emptyset\} : A \mapsto R(A) \text{ such that } R(A) \subseteq A.$$

Rationality axioms We call a rejection function R on $\mathscr{Q}$ **coherent** if for all A, A_1 and A_2 in $\mathscr{Q}$, all f and g in $\mathscr{L}$, and all λ in $\mathbb{R}_{>0}$:

- R_1. $R(A) \neq A$; **[avoiding complete rejection]**
- R_2. if $f < g$ then $f \in R(\{f,g\})$; **[dominance]**
- R_3. a. if $A_1 \subseteq R(A_2)$ and $A_2 \subseteq A$ then $A_1 \subseteq R(A)$; **[Sen's α]**
  b. if $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$ then $A_1 \setminus A \subseteq R(A_2 \setminus A)$; **[Aizerman]**
- R_4. a. if $A_1 \subseteq R(A_2)$ then $\lambda A_1 \subseteq R(\lambda A_2)$; **[scaling invariance]**
  b. if $A_1 \subseteq R(A_2)$ then $A_1 + \{f\} \subseteq R(A_2 + \{f\})$. **[independence]**

We collect all coherent rejection functions in the set $\overline{\mathscr{R}}$.

The 'is not more informative than' relation Given two rejection functions R_1 and R_2:

$$R_1 \text{ is not more informative than } R_2 \Leftrightarrow (\forall A \in \mathscr{Q})(R_1(A) \subseteq R_2(A)).$$

For any collection $\mathbf{R}$ of rejection functions, its infimum is the rejection function given by

$$(\inf \mathbf{R})(A) := \bigcap \mathbf{R}(A) \text{ for all } A \text{ in } \mathscr{Q}.$$

If $\mathbf{R}$ consists of coherent rejection functions, then $\inf \mathbf{R}$ is coherent itself.

Assessment Mostly, if a subject assesses his rejection functions, he will only provide an **incomplete specification**. He will state

"I assess $f \in R(B)$ for some B in $\mathscr{Q}$ and f in B."

or, if we assume that this assessment satisfies Axiom R_4b, equivalently:

"I assess $0 \in R(B)$ for some B in $\mathscr{Q}^0 := \{A \in \mathscr{Q} : 0 \in A\}$."

Formally, his assessment $\mathscr{B}$ is a subset of $\mathscr{Q}^0$:

Assessing $\mathscr{B} \subseteq \mathscr{Q}^0$ means: "**my rejection function satisfies** $(\forall B \in \mathscr{B}) 0 \in R(B)$".

Extending an assessment Given any assessment $\mathscr{B} \subseteq \mathscr{Q}^0$ and any rejection function R on $\mathscr{Q}$, we say that R **extends** the assessment $\mathscr{B}$ if $B \in \mathscr{B} \Rightarrow 0 \in R(B)$ for every B in $\mathscr{Q}$.

Natural extension

Definition Given any assessment $\mathscr{B} \subseteq \mathscr{Q}^0$, the **natural extension** of $\mathscr{B}$ is the rejection function

$$\mathscr{E}(\mathscr{B}) := \inf\{R \in \overline{\mathscr{R}} : (\forall B \in \mathscr{B}) 0 \in R(B)\},$$

where we let $\inf \emptyset$ be equal to $\mathrm{id}_{\mathscr{Q}}$, the identity rejection function that maps every option set to itself.
A special rejection function The definition above is not so useful: it provides no explicit expression. To remedy this, consider the **special rejection function** $R_{\mathscr{B}}$ defined as:

$$R_{\mathscr{B}}(A) := \Big\{ f \in A : (\exists A' \in \mathscr{Q}) \Big(A' \supseteq A \text{ and } (\forall g \in \{f\} \cup A' \setminus A) \\ (A' - \{g\} \cap \mathscr{L}_{>0} \neq \emptyset \text{ or } (\exists B \in \mathscr{B}, \exists \mu \in \mathbb{R}_{>0}) \{g\} + \mu B \preccurlyeq A') \Big) \Big\}$$

for all A in $\mathscr{Q}$, where we define $\preccurlyeq$ on $\mathscr{Q}$ as:

$$A_1 \preccurlyeq A_2 \Leftrightarrow (\forall f_1 \in A_1)(\exists f_2 \in A_2) f_1 \leq f_2 \text{ for all } A_1 \text{ and } A_2 \text{ in } \mathscr{Q}.$$

Assessments avoiding complete rejection We say that $\mathscr{B} \subseteq \mathscr{Q}^0$ **avoids complete rejection** when $R_{\mathscr{B}}$ satisfies Axiom R_1.
Theorem 1. *Consider any assessment $\mathscr{B} \subseteq \mathscr{Q}^0$. Then the following statements are equivalent:*
(i) *$\mathscr{B}$ avoids complete rejection;*
(ii) *There is a coherent extension of $\mathscr{B}$: $(\exists R \in \overline{\mathscr{R}})(\forall B \in \mathscr{B}) 0 \in R(B)$;*
(iii) *$\mathscr{E}(\mathscr{B}) \neq \mathrm{id}_{\mathscr{Q}}$;*
(iv) *$\mathscr{E}(\mathscr{B}) \in \overline{\mathscr{R}}$;*
(v) *$\mathscr{E}(\mathscr{B})$ is the least informative rejection function that is coherent and extends $\mathscr{B}$.*
When any (and hence all) of these equivalent statements hold, then $\mathscr{E}(\mathscr{B}) = R_{\mathscr{B}}$.

Application: purely binary assessments

Assume that the assessment $\mathscr{B} \subseteq \mathscr{Q}^0$ consist of only **binary sets**: $\mathscr{B} \subseteq \{\{0, f\} : f \in \mathscr{L}\}$. Therefore, $B := \bigcap \mathscr{B} \setminus \{0\} \subseteq \mathscr{L}$ is its corresponding **desirability assessment**.

Avoiding non-positivity Given any desirability assessment $B \subseteq \mathscr{L}$, we say that B **avoids non-positivity** when $\mathrm{posi}(B) \cap \mathscr{L}_{\leq 0} = \emptyset$.

The inference mechanism for choice functions has the inference mechanism for desirability as a special case:

Theorem 2. *Consider any purely binary assessment $\mathscr{B} \subseteq \mathscr{Q}^0$. Then $B := \bigcap \mathscr{B} \setminus \{0\} \subseteq \mathscr{L}$ avoids non-positivity if and only if $\mathscr{B}$ avoids complete rejection, and if this is the case, then $\mathscr{E}(\mathscr{B}) = R_{\mathrm{posi}(\mathscr{L}_{>0} \cup B)}$.*

Proposition. *Consider $\mathscr{B} \subseteq \mathscr{Q}^0$. If there is a coherent set of desirable gambles D such that $(\forall B \in \mathscr{B}) B \cap D \neq \emptyset$, then $\mathscr{B}$ avoids complete rejection.*

Therefore, this is a sufficient condition for avoiding complete rejection that is easy to check.

Binary choice

More-than-binary choice Rejection functions are **more-than-binary comparisons** of gambles. Given any rejection function R, we can summarise its binary behaviour in

$$D_R := \{f \in \mathscr{L} : 0 \in R(0, f)\};$$

if R is coherent, then D_R is a coherent set of desirable gambles.

Binary choice There might be multiple rejection functions associated to D; the least informative one is

$$R_D(A) := \{f \in A : (\exists g \in A) g - f \in D\}$$

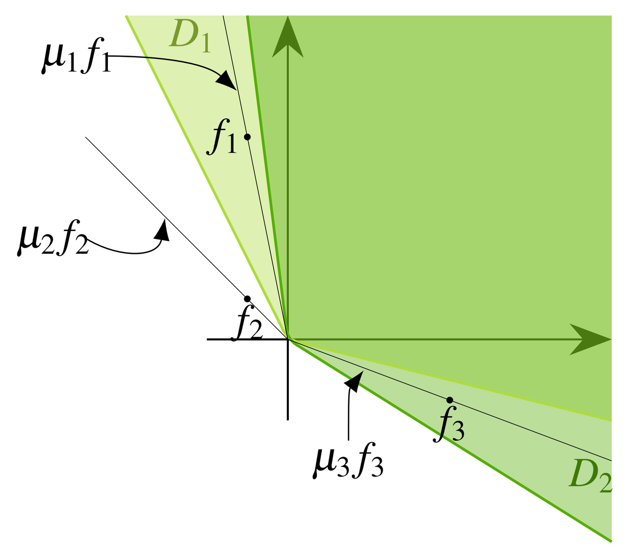
for all A in $\mathscr{Q}$. If D is coherent, then so is R_D. For any collection $\mathscr{D}'$ of coherent sets of desirable gambles, we let $R_{\mathscr{D}'} := \inf\{R_D : D \in \mathscr{D}'\}$. Then

$$0 \in R_{\mathscr{D}'}(A \cup \{0\}) \Leftrightarrow (\forall D \in \mathscr{D}') D \cap A \neq \emptyset$$

for all A in $\mathscr{Q}$.
The natural extension of a desirability assessment $B \subseteq \mathscr{L}$ that avoids non-positivity, is

$$\mathscr{E}_{\mathscr{D}}(B) := \mathrm{posi}(\mathscr{L}_{>0} \cup B).$$

Example: infimum of binary choice

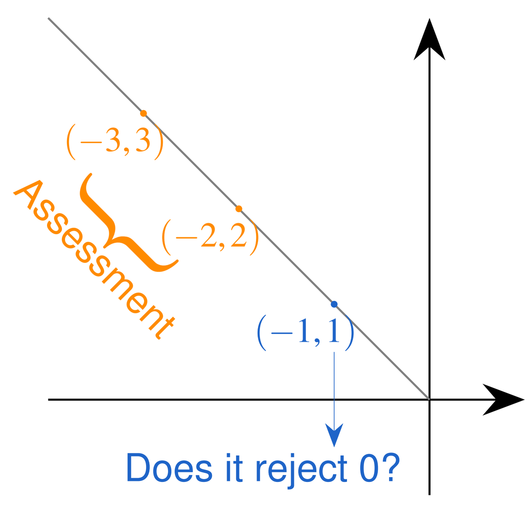


$\mathscr{D}' := \{D_1, D_2\}$ and $A := \{0, f_1, f_2, f_3\}$, so clearly $0 \in R_{\mathscr{D}'}(A)$, since $f_1 \in D_1$ and $f_3 \in D_2$.

Proposition. *Consider any collection $\mathscr{D}'$ of coherent sets of desirable gambles, any $f_1, \dots, f_n$ in $\mathscr{L}$, and any $\mu_1 > 0, \dots, \mu_n > 0$. Then*

$$0 \in R_{\mathscr{D}'}(\{0, f_1, \dots, f_n\}) \\ \Leftrightarrow 0 \in R_{\mathscr{D}'}(\{0, \mu_1 f_1, \dots, \mu_n f_n\}).$$

Example: intrinsic non-binary choice



Assessment Consider the single assessment

$$\mathscr{B} := \{B\} \text{ where } B := \{0, (-2,2), (-3,3)\}.$$

It avoids complete rejection, by the Proposition in the frame **Application: purely binary assessments**. Therefore, $R_{\mathscr{B}}$ is a **coherent** rejection function.
Intrinsic non-binary choice
Note that $0 \in R_{\mathscr{B}}(0, (-2,2), (-3,3))$. We find that

$$0 \notin R_{\mathscr{B}}(\{0, (-1,1)\}) = R_{\mathscr{B}}(\{0, 1/2(-2,2), 1/3(-3,3)\})!$$

It is no infimum of purely binary rejection functions.

Weak extension

Setting We have two random variables X and Y, taking values in the finite possibility spaces $\mathscr{X}$ and $\mathscr{Y}$ respectively. From here on, the set of all gambles on $\mathscr{X} \times \mathscr{Y}$ is denoted by $\mathscr{L}$. This is heavily inspired on [*Gert de Cooman & Enrique Miranda, Irrelevant and independent natural extension for sets of desirable gambles*].
Gambles: cylindrical extension Let f be a gamble on $\mathscr{X}$. Define its **cylindrical extension** f^*:

$$f^*(x,y) := f(x) \text{ for all } (x,y) \text{ in } \mathscr{X} \times \mathscr{Y}.$$

f^* belongs to $\mathscr{L}$. Similarly, for any set A of gambles on $\mathscr{X}$, we let $A^* := \{f^* : f \in A\}$.
Marginalisation Consider any rejection function R on $\mathscr{X} \times \mathscr{Y}$. Define its **$X$-marginal** $\mathrm{marg}_X(R)$ as

$$(\mathrm{marg}_X(R))(A) := R(A^*) \text{ for all } A \text{ in } \mathscr{Q}(\mathscr{X}).$$

If R is coherent, then so is $\mathrm{marg}_X(R)$.
Rejection function: weak extension Let R be a coherent rejection function on $\mathscr{X}$.

What is the least informative coherent rejection function on $\mathscr{X} \times \mathscr{Y}$ that marginalises to R?

Proposition. *The least informative coherent rejection function on $\mathscr{X} \times \mathscr{Y}$ that marginalises to R is $R_{\mathscr{A}}$, where*

$$\mathscr{A} := \{A^* : A \in \mathscr{Q}^0(\mathscr{L}(\mathscr{X})), 0 \in R(A)\}.$$

$R_{\mathscr{A}}$ is called the **weak extension** *of R.*

Irrelevant natural extension

Conditioning Consider any rejection function R on $\mathscr{X} \times \mathscr{Y}$. For every y in $\mathscr{Y}$, define its **conditioned rejection function** $R\rfloor y$ on $\mathscr{X}$ as

$$R\rfloor y(A) := \{f \in A : \mathbb{I}_{\{y\}} f \in R(\mathbb{I}_{\{y\}} A)\} \text{ for all } A \text{ in } \mathscr{Q}(\mathscr{X}),$$

where we let $\mathbb{I}_{\{y\}} := \{\mathbb{I}_{\{y\}} f : f \in A\}$ be a set of gambles on $\mathscr{X} \times \mathscr{Y}$. If R is coherent, then so is $R\rfloor y$.
Epistemic irrelevance We say that X is **epistemic irrelevant** to Y when learning the value of X does not influence our beliefs about Y. A rejection function R on $\mathscr{X} \times \mathscr{Y}$ **satisfies epistemic irrelevance** of X to Y when $\mathrm{marg}_Y(R\rfloor x) = \mathrm{marg}_Y(R)$ for all x in $\mathscr{X}$
Proposition. *Consider any coherent rejection function R on $\mathscr{X} \times \mathscr{Y}$. Then R satisfies epistemic irrelevance of X to Y if and only if*

$$(\forall A \in \mathscr{Q}(\mathscr{X}))(\forall y \in \mathscr{Y}) 0 \in R(A) \Leftrightarrow 0 \in R(\mathbb{I}_{\{y\}} A).$$

Let R be a coherent rejection function on $\mathscr{X}$.

What is the least informative coherent rejection function on $\mathscr{X} \times \mathscr{Y}$ that marginalises to R and satisfies epistemic irrelevance from X to Y?

Theorem 3. *The least informative rejection function on $\mathscr{X} \times \mathscr{Y}$ that marginalises to R and satisfies epistemic irrelevance of X to Y is $R_{\mathscr{A}_{X \to Y}}$, where*

$$\mathscr{A}_{X \to Y} := \{\mathbb{I}_{\{y\}} A : A \in \mathscr{Q}(\mathscr{X}), 0 \in R(A), y \in \mathscr{Y}\} \cup \{A^* : A \in \mathscr{Q}(\mathscr{X}), 0 \in R(A)\}.$$