imprecise probabilities can have a clear empirical/frequentist meaning only if they can be estimated from data.

consider for example a (potentially infinite) sequence of bags containing only white and black marbles: we draw one marble at random from each bag, where the proportion of black marbles in the $i$-th bag is $p_i \in [\underline{p}, \bar{p}] \subseteq [0, 1]$. 

if $\underline{p} = \bar{p}$, then $[\underline{p}, \bar{p}]$ represents a precise probability ($P$), which can be estimated from data without problems (Bernoulli, 1713).

if $\underline{p} < \bar{p}$, then $[\underline{p}, \bar{p}]$ represents an imprecise probability (IP): can it still be estimated from data?
interpretations of $[p, \bar{p}]$

- which sequences of proportions $p_i$ are compatible with the IP $[p, \bar{p}]$?

- **epistemological** indeterminacy interpretation (Walley and Fine, 1982), used e.g. in the theory of Markov chains with IPs (Kozine and Utkin, 2002):
  \[ p_i = p \in [p, \bar{p}] \]

- **ontological** indeterminacy interpretation (Walley and Fine, 1982), used e.g. in the theories of Markov chains with IPs (Hartfiel, 1998) and probabilistic graphical models with IPs (Cozman, 2005):
  \[ p_i \in [p, \bar{p}] \]

- **id-ontological** (identifiable ontological indeterminacy interpretation), making $[p, \bar{p}]$ identifiable:
  \[ p_i \in [p, \bar{p}] = \left[ \liminf_{i \to \infty} p_i, \limsup_{i \to \infty} p_i \right] \]
levels of estimability of $[\underline{p}, \overline{p}]$

- assuming that we have a sufficiently large number $n$ of drawings

- **ideal:** uniformly consistent estimability, meaning that we can construct arbitrarily short confidence intervals for $\underline{p}$ and $\overline{p}$ with arbitrarily high confidence levels

- **minimal:** IP-consistent estimability (i.e. consistent in terms of IPs), called strong estimability by Walley and Fine (1982), and almost equivalent to the testability of $[\underline{p}, \overline{p}]$ with arbitrarily low significance level and arbitrarily high power for a fixed alternative

- **inadequate:** $P$-consistent estimability (i.e. consistent in terms of $Ps$), meaning that $\underline{p}$ and $\overline{p}$ can be estimated arbitrarily well under each compatible sequence $p_i$, but the level of precision of the estimator can depend on the particular sequence $p_i$
estimability of $[\underline{p}, \overline{p}]$

<table>
<thead>
<tr>
<th>necessary and sufficient conditions on possible $[\underline{p}, \overline{p}]$:</th>
<th>epistemological: $p_i = p \in [\underline{p}, \overline{p}]$</th>
<th>ontological: $p_i \in [\underline{p}, \overline{p}]$</th>
<th>id-ontological: $p_i \in [\underline{p}, \overline{p}]$ s.t. $p = \lim \inf_{i \to \infty} p_i$, $\overline{p} = \lim \sup_{i \to \infty} p_i$</th>
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<tr>
<td><strong>ideal:</strong> uniformly consistent</td>
<td>pairwise disjoint and IPs isolated</td>
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estimability of \( [\min\{p_1, \ldots, p_n\}, \max\{p_1, \ldots, p_n\}] \)

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<th>necessary and sufficient conditions on possible ([p, \bar{p}]):</th>
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<tr>
<td>ideal: uniformly consistent</td>
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conclusion

- IPs $[p, \bar{p}]$ can be empirically distinguished only if they are disjoint.
- Finite-sample IPs $[\min\{p_1, \ldots, p_n\}, \max\{p_1, \ldots, p_n\}]$ cannot be estimated from data.
- The paper summarizes several results that are not surprising, but important to clarify the limited empirical/frequentist meaning of IPs.
- Examples of estimators with the required properties are given in the paper.
J. Bernoulli. *Ars Conjectandi*. Thurneysen Brothers, 1713.


