Online Algorithms for Sum-Product Networks with Continuous Variables

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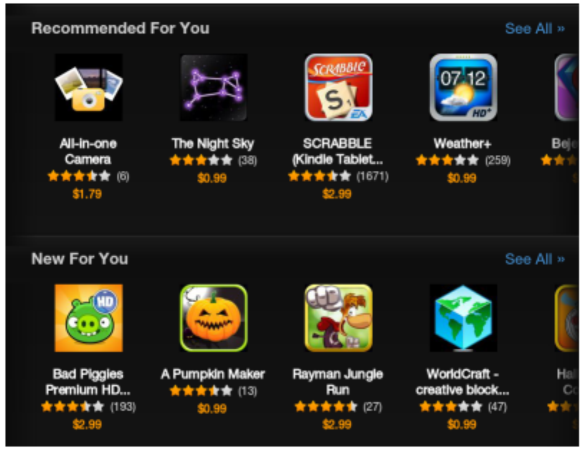
Streaming Numerical Data

Traffic classification

App recommendation



- **Challenge:** update model after each data vector
- **Solution:** online learning for continuous SPNs

Outline

- Background: Sum product networks
- Gaussian Sum-Product Networks
  - Hierarchical mixture of Gaussians
  - Online algorithm: Bayesian Moment Matching
- Experiments: comparison with
  - Other online algorithms: oEM, SGD
  - Other generative models: stacked RBMs, GenMMNs
- Conclusion and future work

What is a Sum-Product Network?

- Prosposed by Poon and Domingos (UAI 2011) (equivalent to arithmetic circuit, Darwiche 2003)

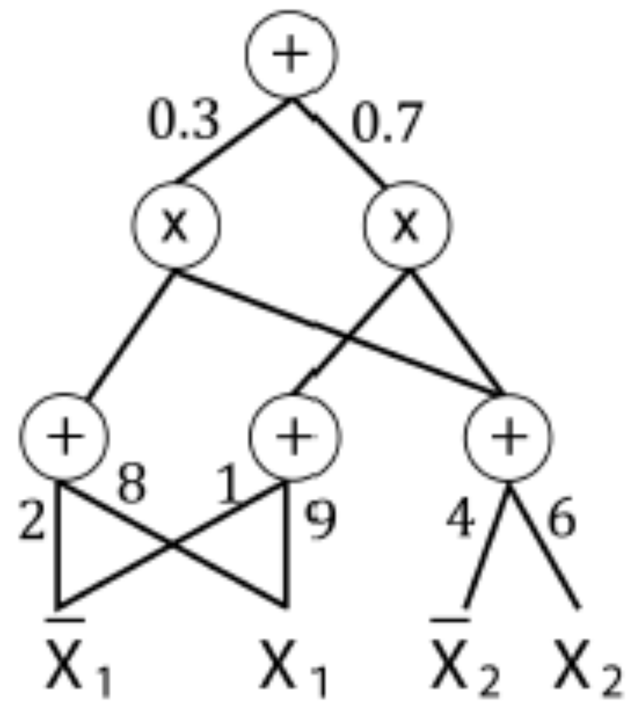
- Two views:

Deep architecture with clear semantics	Tractable probabilistic graphical model

Deep Architecture

- Specific type of deep neural network
  - Sum node: $\log(\sum_{i} w_{i}\, in$
  - Product node: $\exp(\sum_{i} inp$

- Advantages:
  - Clear semantics
  - Generative models
  - Flexible queries



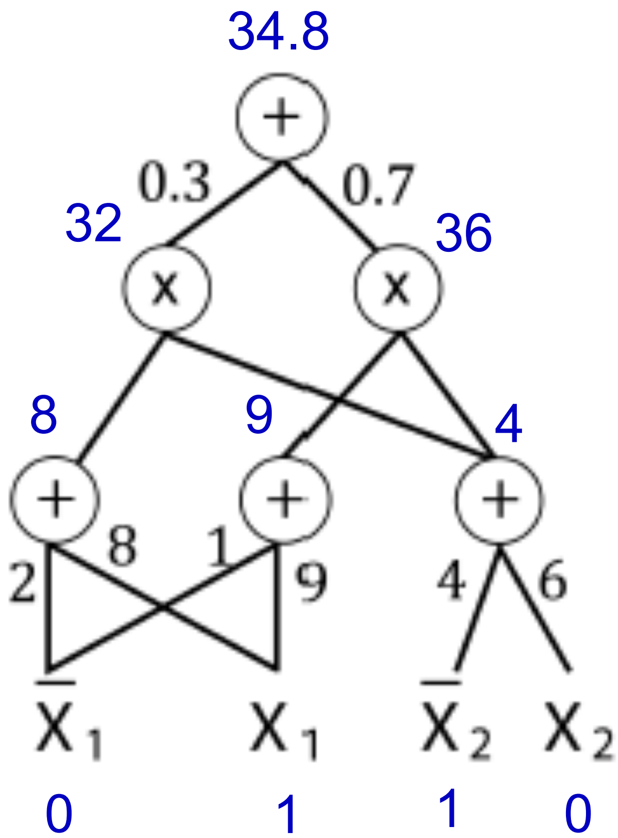
Probabilistic Inference

- SPN represents a joint distribution over a set of random variables

- Example:

$\Pr(X{\downarrow}1 = true, X{\downarrow}2 = fals$

$= 34.8/$

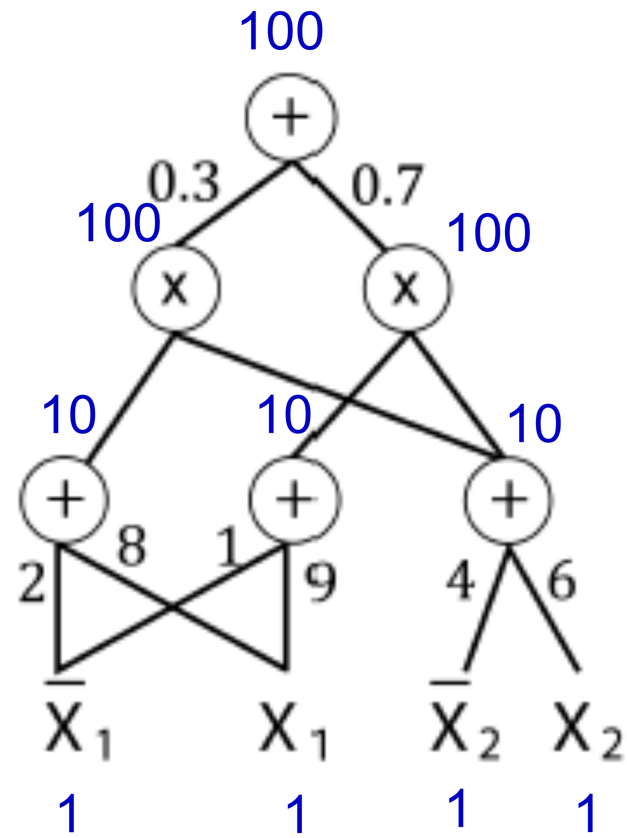


Probabilistic Inference

- SPN represents a joint distribution over a set of random variables

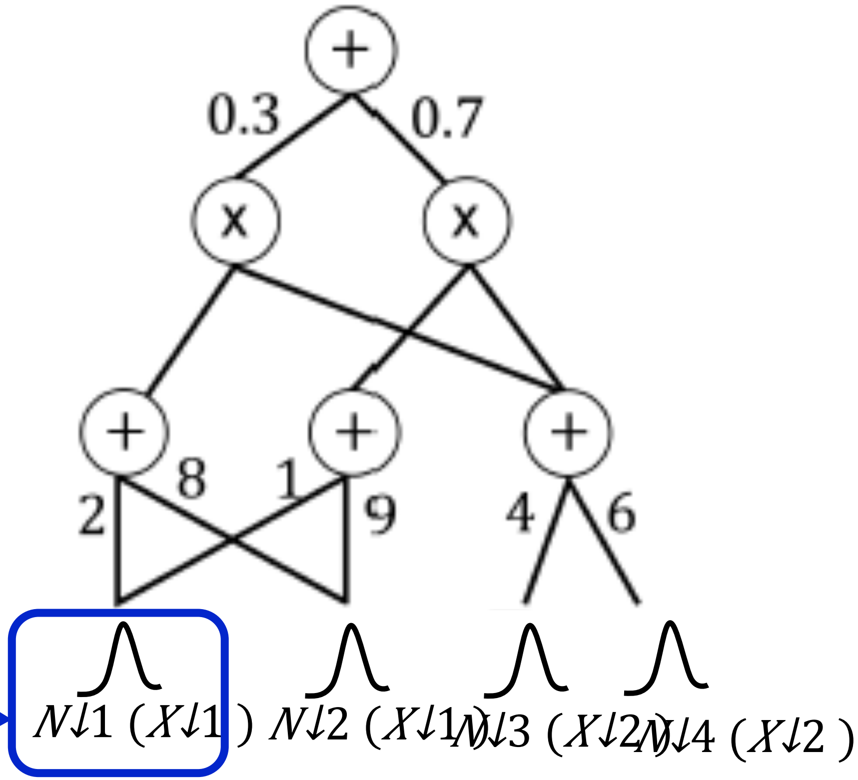
- Example:
$\Pr(X{\downarrow}1 = true, X{\downarrow}2 = fals$
$= 34.8/$
100

- **Linear complexity!**



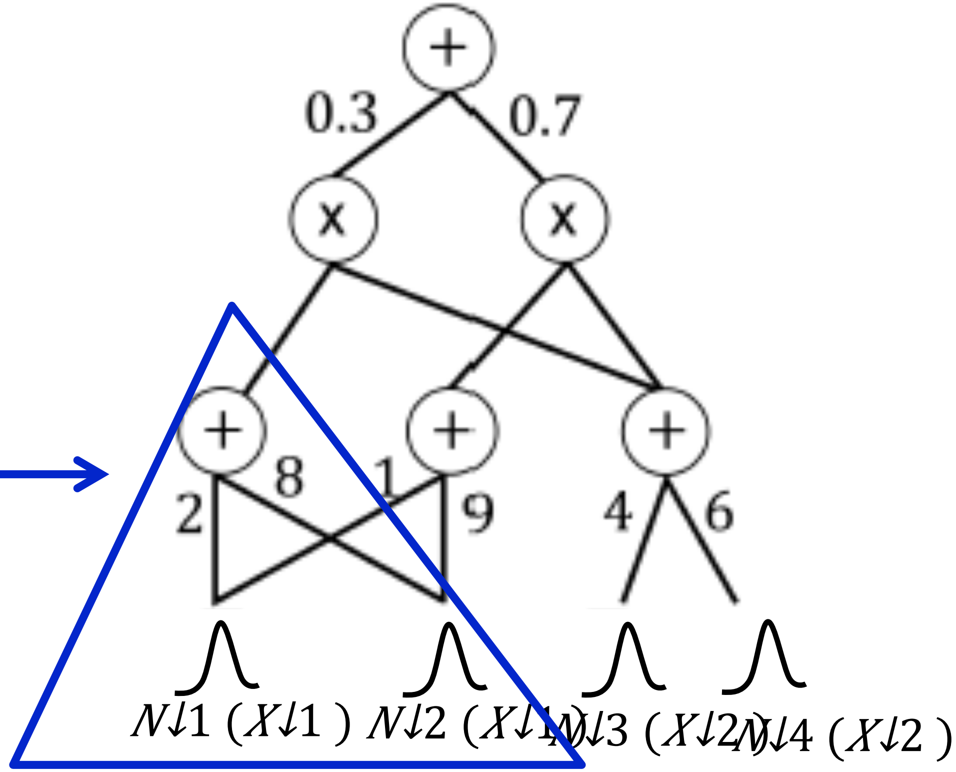
Continuous SPNs

Gaussian



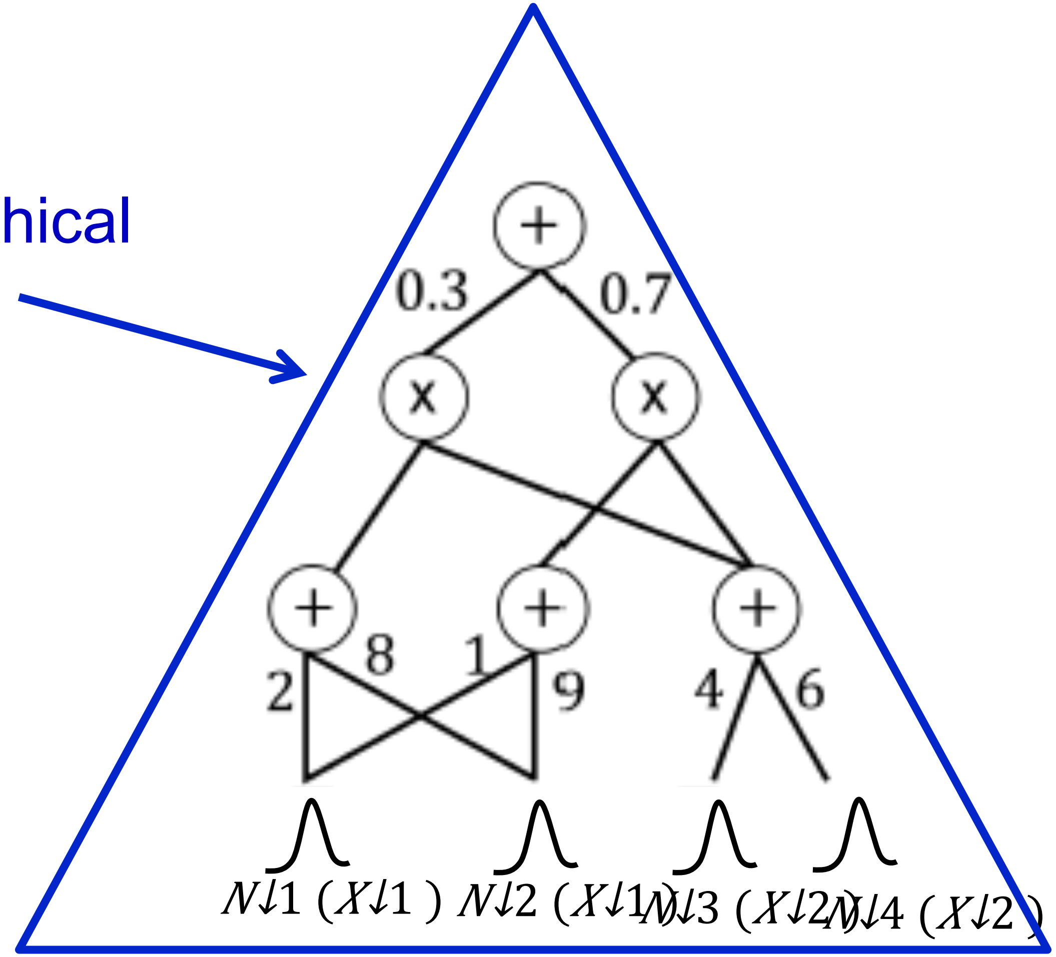
Continuous SPNs

(Unnormalized)
mixture of Gaussians



Continuous SPNs

(Unnormalized) hierarchical mixture of Gaussians



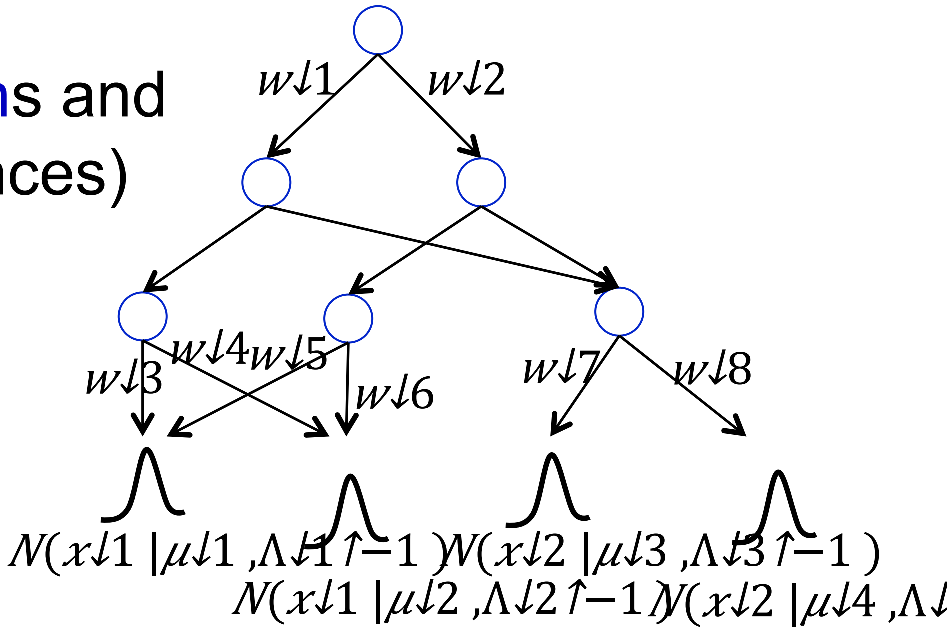
Learning SPNs

- Structure Learning by
  - **Clustering** (Dennis and Ventura, 2012; Gens and Domingos, 2013; Peharz et al., 2013; Rooshenas and Lowd, 2014; Adel et al., 2015; Vergari et al., 2015)
- Parameter Learning by
  - **Maximum likelihood**: stochastic gradient descent (SGD) (Poon & Domingos, 2011), expectation maximization (EM) (Perharz, 2015), signomial programming (Zhao & Poupart, 2016)
  - **Bayesian learning**: Bayesian Moment Matching (BMM) (Rashwan et al., 2015), Collapsed Variational Inference (Zhao et al., 2016)

Streaming Data

- Online parameter learning for discrete SPNs
  - Rashwan, Zhao and Poupart, AISTATS-2016:
    - **Stochastic Gradient Descent:** slow convergence, inaccurate (parameters: step size, mini-batch size, learning rate)
    - **Online Expectation Maximization (oEM):** inaccurate (parameters: mini-batch size, learning rate)
    - **Online Bayesian Moment Matching (oBMM): accurate**
  - Zhao, Adel, Gordon and Amos, ICML-2016:
    - **Collapsed Variational Inference: accurate**
- Online parameter learning for continuous SPNs
  - Contribution: **extend oBMM to Gaussian SPNs**

Bayesian Learning



- **Parameters**: weights, means and precisions (inverse covariances)
- WLOG consider normalized SPNs (normalized weights)
- **Prior**: $P(\boldsymbol{w}, \boldsymbol{\mu}, \boldsymbol{\Lambda})$
  product of Dirichlets and Normal-Wisharts
- **Likelihood:** $P \boldsymbol{x} \boldsymbol{w}, \boldsymbol{\mu}, \boldsymbol{\Lambda} = SPN(\boldsymbol{x}; \boldsymbol{w}, \boldsymbol{\mu}, \boldsymbol{\Lambda})$

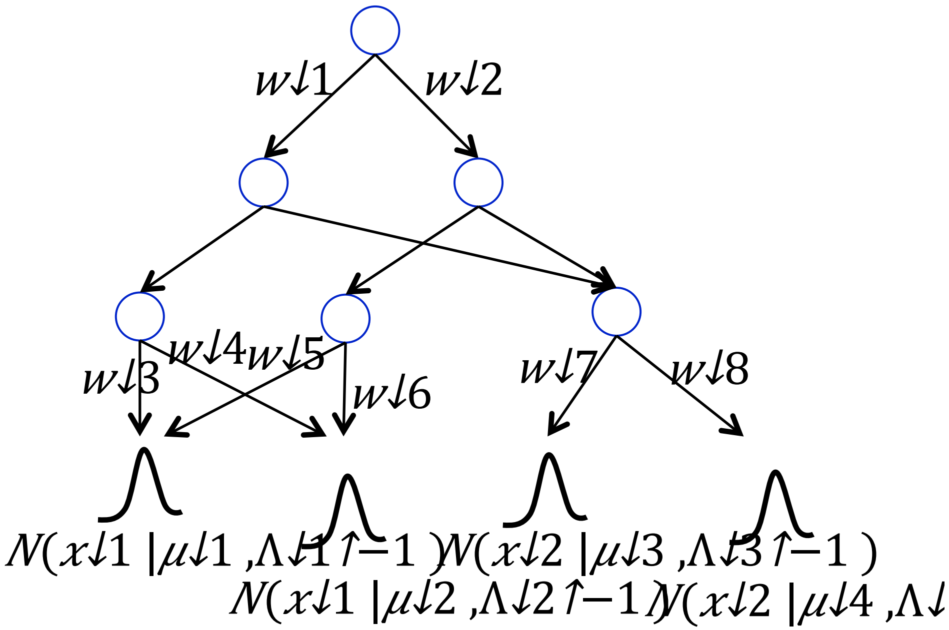
Bayesian Learning

- **Posterior**: $P\,\boldsymbol{w},\boldsymbol{\mu},\boldsymbol{\Lambda}\,data$

$\propto P(\boldsymbol{w},\boldsymbol{\mu},\boldsymbol{\Lambda})\; SPN\,\boldsymbol{x}^{(1)}\;\boldsymbol{w},\boldsymbol{\mu},\boldsymbol{\Lambda}\; SPN\,\boldsymbol{x}^{(2)}\;\boldsymbol{w},\boldsymbol{\mu},\boldsymbol{\Lambda}$
$\ldots SPN\,\boldsymbol{x}^{(n)}\;\boldsymbol{w},\boldsymbol{\mu},\boldsymbol{\Lambda}$

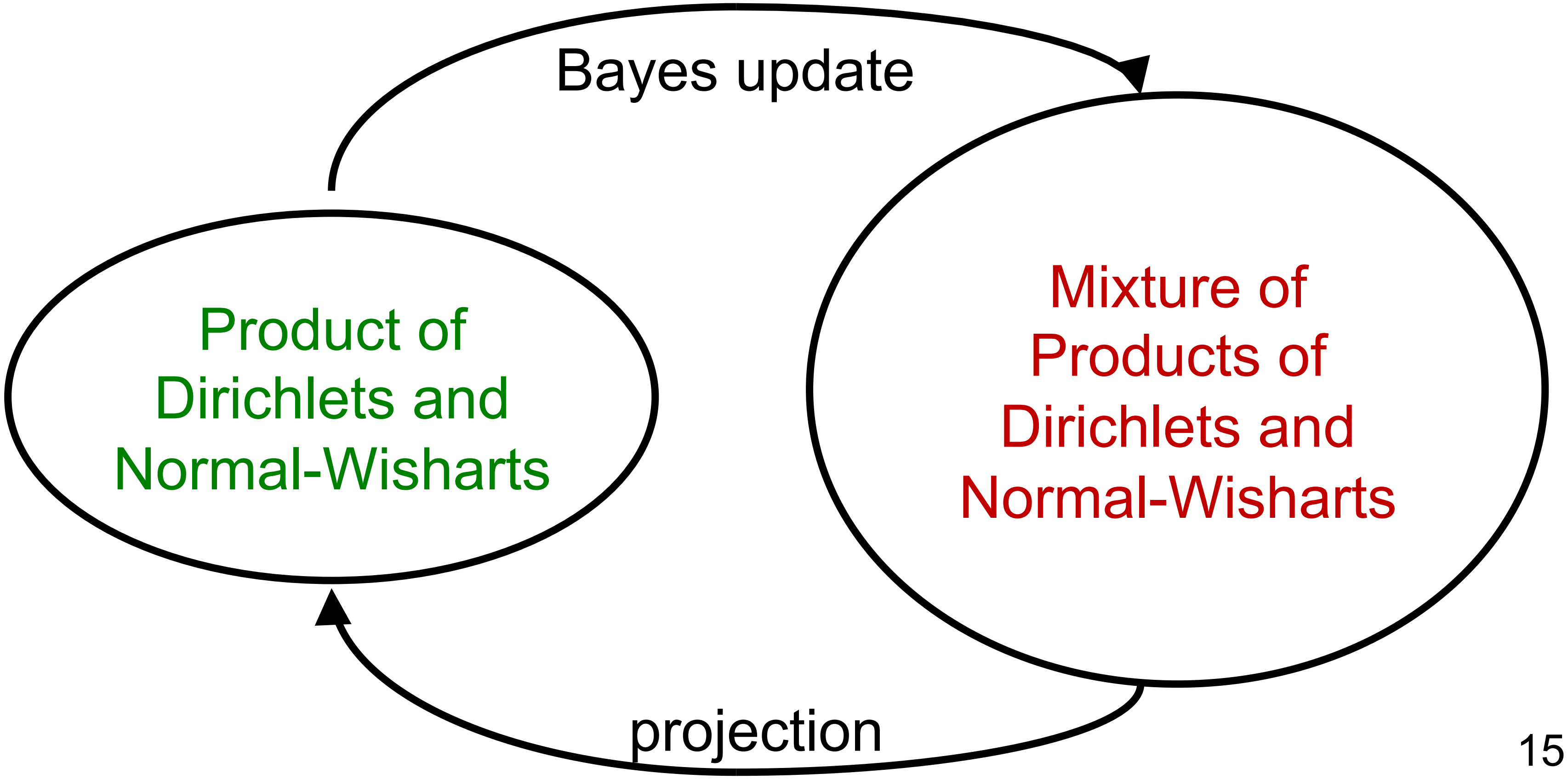
- **Naturally online learning!**

- **Problem: intractable** exponentially large mixture of products of Dirichlets and Normal-Wisharts



Moment Matching

- Solution: project posterior in a tractable family (assumed density filtering)



Sufficient Moments

- Dirichlet: $Dir(w_1,\dots,w_M;\alpha_1,\dots,\alpha_M)$

  $E[w_i]=\alpha_i/\sum_j \alpha_j \qquad E[w_i^2]=\alpha_i(\alpha_i+1)/(\sum_j \alpha_j)(1+\sum_j \alpha_j)$

- Normal-Wishart: $NW(\boldsymbol{\mu},\boldsymbol{\Lambda};\boldsymbol{\mu}_0,\kappa,\boldsymbol{W},\nu)$

  $E[\boldsymbol{\mu}]=\boldsymbol{\mu}_0$

  $E[(\boldsymbol{\mu}-\boldsymbol{\mu}_0)(\boldsymbol{\mu}-\boldsymbol{\mu}_0)^T]=\kappa+1/\kappa(\nu-d-1)\,\boldsymbol{W}^{-1}$

  $E[\boldsymbol{\Lambda}]=\nu\boldsymbol{W}$

  $Var[\Lambda_{ij}]=\nu(W_{ij}^2+W_{ii}W_{jj})$

Posterior Moments

- Moment of some parameter s is

$$E[s]=\int_s s P(s|x)\, ds$$

- At leaf node i, for each $s \in \{\boldsymbol{\mu}_i, \boldsymbol{\mu}_i \boldsymbol{\mu}_i^T, \boldsymbol{\Lambda}_i, \Lambda_{ijk}^2\}$:

$$E[s]=\int_s s\, NW(\boldsymbol{\mu}_i, \boldsymbol{\Lambda}_i | \boldsymbol{\alpha}_i, \kappa_i, \boldsymbol{W}_i, \nu_i)\left(c_i^0 + c_i^1 N(\boldsymbol{x}|\boldsymbol{\mu}_i, \boldsymbol{\Lambda}_i^{-1})\right)ds$$

- At sum node i:

$$E[w_{ij}^k]=\int_{\boldsymbol{w}_{i\cdot}} w_{ij}^k\, Dir(\boldsymbol{w}_{i\cdot} | \boldsymbol{\alpha}_{i\cdot})\left(c_i^0 + c_i^1 \sum_{j'} w_{ij'} V_{j'}(\boldsymbol{x})\right) d\boldsymbol{w}_{i\cdot}$$

Overall Algorithm

- Recursive computation of all c_i^0 and c_i^1
  - Two passes (details in paper)
  - Linear complexity in size of SPN
- Moment matching
  - System of linear equations
  - Linear complexity in size of SPN
- Streaming data
  - Posterior update: constant time w.r.t. amount of data

Empirical Results

Log likelihood and standard error based on 10-fold cross validation

Dataset	Flow Size	Quake	Banknote	Abalone	Kinematics	CA	Sensorless Drive
# of vars	3	4	4	8	8	22	48
oBMM (random)	-	-	-	**-1.82** $\pm$ 0.19	**-11.19** $\pm$ 0.03	**-2.47** $\pm$ 0.56	**1.58** $\pm$ 1.28
oEM (random)	-	-	-	-11.36 $\pm$ 0.19	-11.35 $\pm$ 0.03	-31.34 $\pm$ 1.07	-3.40 $\pm$ 6.06
oBMM (GMM)	**4.80** $\pm$ 0.67	**-3.84** $\pm$ 0.16	**-4.81** $\pm$ 0.13	**-1.21** $\pm$ 0.36	**-11.24** $\pm$ 0.04	**-1.78** $\pm$ 0.59	-
oEM (GMM)	-0.49 $\pm$ 3.29	-5.50 $\pm$ 0.41	-4.81 $\pm$ 0.13	-3.53 $\pm$ 1.68	-11.35 $\pm$ 0.03	-21.39 $\pm$ 1.58	-

oBMM more accurate than oEM

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SBRM	-0.79 ± 0.01	**-2.38** ± 0.01	-2.76 ± 0.01	-2.28 ± 0.01	**-5.55** ± 0.02	-4.95 ± 0.01	-26.91 ± 0.03
GenMMN	0.40 ± 0.01	-3.83 ±	**-1.70** ± 0.03	-3.29 ± 0.10	-11.36 ± 0.02	-5.41 ± 0.14	-29.41 ±1.19

oBMM competitive with SBRM and GenMMN

Conclusion

- Contributions
  - Continuous SPNs with Gaussian leaves
  - Online Bayesian Moment Matching for streaming data

- Future work
  - More extensive experiments with larger problems
  - Online structure learning
  - Generalize leaf distributions to exponential family