Grothendieck toposes as unifying 'bridges' in Mathematics

Olivia Caramello

Università degli Studi dell'Insubria (Como)

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The "unifying notion" of topos

"It is the topos theme which is this "bed" or "deep river" where come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the "continuous" and that of "discontinuous" or discrete structures. It is what I have conceived of most broad to perceive with finesse, by the same language rich of geometric resonances, an "essence" which is common to situations most distant from each other coming from one region or another of the vast universe of mathematical things".

A. Grothendieck

Topos theory can be regarded as a unifying subject in Mathematics, with great relevance as a framework for systematically investigating the relationships between different mathematical theories and studying them by means of a multiplicity of different points of view. Its methods are transversal to the various fields and complementary to their own specialized techniques. In spite of their generality, the topos-theoretic techniques are liable to generate insights which would be hardly attainable otherwise and to establish deep connections that allow effective transfers of knowledge between different contexts.

The multifaceted nature of toposes

The role of toposes as unifying spaces is intimately tied to their multifaceted nature.

For instance, a topos can be seen as:

- a generalized space
- a mathematical universe
- a theory modulo 'Morita-equivalence'

We shall now review each of these different points of view.

Toposes as generalized spaces

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a pair $(\mathscr{C}, J)$ consisting of a (small) category $\mathscr{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the pair:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Sh}(X) \\ \wr\!\downarrow & & \wr\!\downarrow \\ (\mathscr{C}, J) & \dashrightarrow & \mathbf{Sh}(\mathscr{C}, J) \end{array}$$

Topos-theoretic invariants

- The notion of a geometric morphism of toposes has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'.
  The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

Toposes as mathematical universes

A decade later, W. Lawvere and M. Tierney discovered that a topos could not only be seen as a generalized space, but also as a mathematical universe in which one can do mathematics similarly to how one does it in the classical context of sets (with the only important exception that one must argue constructively).

Amongst other things, this discovery made it possible to:

- Exploit the inherent 'flexibility' of the notion of topos to construct 'new mathematical worlds' having particular properties.
- Consider models of any kind of (first-order) mathematical theory not just in the classical set-theoretic setting, but inside every topos, and hence 'relativise' Mathematics.

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Classifying toposes

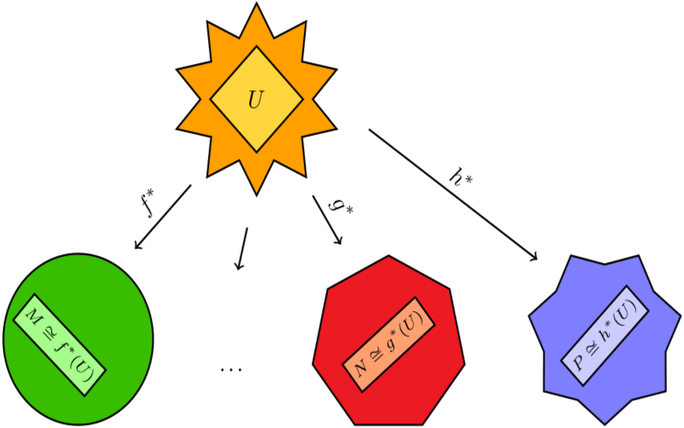
It was realized in the seventies (thanks to the work of several people, notably including W. Lawvere, A. Joyal, G. Reyes and M. Makkai) that:

- To any (geometric first-order) mathematical theory $\mathbb{T}$ one can canonically associate a topos $\mathscr{E}_{\mathbb{T}}$, called the classifying topos of the theory, which represents its 'semantical core'.
- The topos $\mathscr{E}_{\mathbb{T}}$ is characterized by the following representability property: for any Grothendieck topos $\mathscr{E}$ we have an equivalence of categories

$$\textbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}}) \simeq \mathbb{T}\text{-mod}(\mathscr{E})$$

  natural in $\mathscr{E}$, where
  - $\textbf{Geom}(\mathscr{E}, \mathscr{E}_{\mathbb{T}})$ is the category of geometric morphisms $\mathscr{E} \to \mathscr{E}_{\mathbb{T}}$ and
  - $\mathbb{T}\text{-mod}(\mathscr{E})$ is the category of $\mathbb{T}$-models in $\mathscr{E}$.

Classifying toposes



Classifying topos

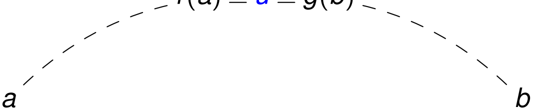
Toposes as theories up to 'Morita-equivalence'

- Two mathematical theories have the same classifying topos (up to equivalence) if and only if they have the same 'semantical core', that is if and only if they are indistinguishable from a semantic point of view; such theories are said to be Morita-equivalent.
- Conversely, every Grothendieck topos arises as the classifying topos of some theory.
- So a topos can be seen as a canonical representative of equivalence classes of theories modulo Morita-equivalence.

The idea of *bridge*

- We can think of a *bridge* object connecting two objects a and b as an object u which can be 'built' from any of the two objects and admits two different representations $f(a)$ and $g(b)$ related by some kind of equivalence $\simeq$, the former being in terms of the object a and the latter in terms of the object b:

$$f(a) \simeq u \simeq g(b)$$



- The transfer of information arises from the process of 'translating' invariant (with respect to $\simeq$) properties of (resp. constructions on) the 'bridge object' u into properties of (resp. constructions on) the two objects a and b by using the two different representations $f(a)$ and $g(b)$ of the bridge object.

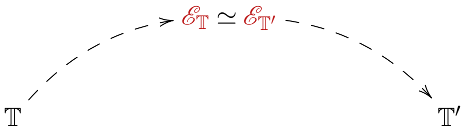
Toposes as *bridges*

- Toposes can effectively act as 'bridge objects' across Morita-equivalent theories.
- The notion of Morita-equivalence is ubiquitous in Mathematics; indeed, it formalizes in many situations the feeling of 'looking at the same thing in different ways', or 'constructing a mathematical object through different methods'.
- In fact, many important dualities and equivalences in Mathematics can be naturally interpreted in terms of Morita-equivalences.
- On the other hand, Topos Theory itself is a primary source of Morita-equivalences. Indeed, different representations of the same topos can be interpreted as Morita-equivalences between different mathematical theories.
- Any two theories which are bi-interpretable in each other are Morita-equivalent but, very importantly, the converse does not hold.
- A mathematical theory alone gives rise to an infinite number of Morita-equivalences, through its 'internal dynamics'.

Toposes as *bridges*

- The existence of different theories with the same classifying topos translates, at the technical level, into the existence of different representations for the same topos.

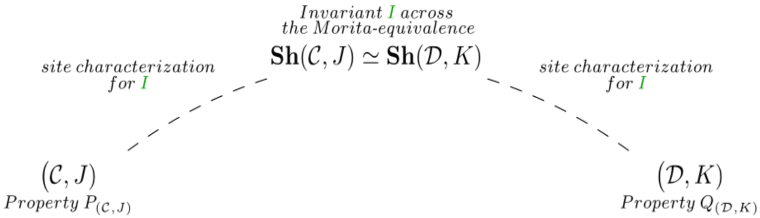
- Topos-theoretic invariants can thus be used to transfer information from one theory to another:



- The transfer of information takes place by expressing a given invariant in terms of the different representations of the topos.
- As such, different properties (resp. constructions) arising in the context of theories classified by the same topos are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

The 'bridge-building' technique

- Decks of 'bridges': Morita-equivalences (or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



The 'bridge' yields a logical equivalence (or an implication) between the 'concrete' properties $P_{(\mathscr{C},J)}$ and $Q_{(\mathscr{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

Some applications

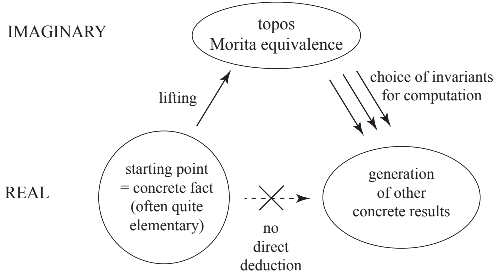
- Model theory (topos-theoretic interpretation and generalization of Fraïssé's theorem)
- Algebra (topos-theoretic generalization of the Galois formalism and unification with the generalized Fraïssé theory)
- Topology (topos-theoretic interpretation/generation of Stone-type dualities)
- Proof theory (new proof systems based on Grothendieck topologies)

A mathematical morphogenesis

- The essential ambiguity given by the fact that any topos is associated in general with an infinite number of theories or different sites allows to study the relations between different theories, and hence the theories themselves, by using toposes as 'bridges' between these different presentations.
- Every topos-theoretic invariant generates a veritable mathematical morphogenesis resuting from its expression in terms of different representations of toposes, which gives rise in general to connections between properties or notions that are completely different and apparently unrelated from each other
- The mathematical exploration is therefore in a sense 'reversed' since it is guided by the Morita-equivalences and by topos-theoretic invariants, from which one proceeds to extract concrete information on the theoires that one wishes to study.

The duality between 'real' and 'imaginary'

- The passage from a site (or a theory) to the associated topos can be regarded as a sort of 'completion' by the addition of 'imaginaries' (in the model-theoretic sense), which materializes the potential contained in the site (or theory).
- The duality between the (relatively) unstructured world of presentations of theories and the maximally strucured world of toposes is of great relevance as, on the one hand, the 'simplicity' and concreteness of theories or sites makes it easy to manipulate them, while, on the other hand, computations are much easier in the 'imaginary' world of toposes thanks to their very rich internal structure and the fact that invariants live at this level.



Some key features of toposes

Here are some essential features of toposes, which account for their relevance in Mathematics:

- Generality: Unlike most of the invariants used in Mathematics, the level of generality of topos-theoretic invariants is such as to make them suitable for effectively comparing with each other theories or objects coming from different fields of Mathematics.
- Expressivity: On the other hand, many important invariants arising in Mathematics can be expressed as topos-theoretic invariants (think for instance of the cohomological and homotopy-theoretic invariants).
- Centrality: The fact that topos-theoretic invariants often manifest as important properties or constructions of natural mathematical or logical interest is a clear indication of the centrality of these concepts in Mathematics. In fact, whatever happens at the level of toposes has 'uniform' ramifications in Mathematics as a whole.
- Technical flexibility: Toposes are mathematical universes which are very rich in terms of internal structure; moreover, they have a very-well behaved representation theory, which makes them extremely effective computational tools, in particular when they are considered as 'bridges'.

Future directions

The evidence provided by the results obtained so far shows that toposes can effectively act as unifying spaces for transferring information between distinct mathematical theories and for generating new equivalences, dualities and symmetries across different fields of Mathematics.

In fact, toposes have an authentic creative power in Mathematics, in the sense that their study naturally leads to the discovery of a great number of notions and 'concrete' results in different mathematical fields, which are pertinent but often unsuspected.

In the next years, we intend to continue pursuing the development of these general unifying methodologies both at the theoretical level and at the applied level, in order to continue developing the potential of toposes as fundamental tools in the study of mathematical theories and their relations, and as key concepts defining a new way of doing Mathematics liable to bring distinctly new insights in a great number of different subjects.

For further reading

- O. Caramello
  Grothendieck toposes as unifying 'bridges' in Mathematics,
  Mémoire d'habilitation à diriger des recherches,
  Université de Paris 7, 2016,
  available from my website www.oliviacaramello.com.

- O. Caramello.
  Theories, Sites, Toposes: Relating and studying mathematical theories through topos-theoretic 'bridges',
  Oxford University Press, 2017.