 $\oplus$  $= QM$

Bayes+Hilbert=Quantum Mechanics

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Hats Superposition

Alessio Benavoli
Alessandro Facchini
Marco Zaffalon

Content of the talk

Subjective foundation of quantum mechanics

This consists in showing that:

- It is possible to derive all axioms (and rules) of QM from a single principle of **self-consistency (rationality)** or, in other words, that QM laws of Nature are logically consistent.
- QM is just the **Bayesian theory generalised to the complex Hilbert space**.

Quantum Bayesianism

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Can Quantum Bayesianism Fix the Paradoxes of Quantum Mechanics?

A new version of quantum theory sweeps away the bizarre paradoxes of the microscopic world. The cost? Quantum information exists only in your imagination

- I. Pitowsky (2003). “Betting on the outcomes of measurements: a Bayesian theory of quantum probability.” Studies in History and Philosophy of Modern Physics 34.3: 395-414.
- C. A. Fuchs, R. Schack (2013). “Quantum-Bayesian coherence.” Reviews of Modern Physics 85: 1693
- C. A. Fuchs, N. D. Mermin, R. Schack (2013). “An introduction to QBism with an application to the locality of quantum mechanics.” American Journal of Physics 82.8 (2014): 749-754.
- N. D. Mermin (2014). “Physics: QBism puts the scientist back into science.” Nature 507.7493: 421-423.

Subjective Foundation of Probability

- B. de Finetti (1937). *La prevision: ses lois logiques, ses sources subjectives*. Annales de l'Institut Henri Poincar é English translation in (Kyburg Jr. and Smokler, 1964).
- P. Williams (1975). "Coherence, strict coherence, and zero probabilities." Proceedings of the Fifth International Congress on Logic, Methodology, and Philosophy of Science, vol. VI Reidel Dordrecht, 29-33.
- P. Walley (1991). *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall.

Argument

$$\frac{\text{Classical probability theory}}{\text{Theory of desirable gambles over real numbers}}$$

$$\sim$$

$$\frac{\text{Quantum mechanics}}{\text{Theory of desirable gambles over complex numbers}}$$

(Classical definition) Theory of probability

1. Probability is a number between 0 and 1.
2. Probability of the certain event is 1.
3. Probability of the event "A or B" is $P(A \vee B) = P(A) + P(B)$ (if the events are mutually exclusive);
4. the conditional probability of B given the event A is defined by Bayes' rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \text{ with } P(A) > 0.$$

From these axioms, it is also possible to derive marginalization, law of total probability, independence and so on.

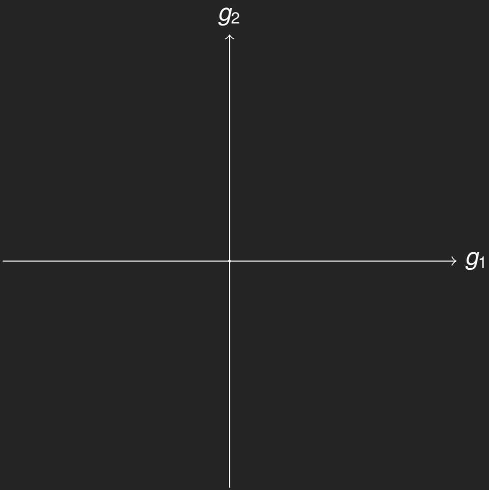
(Classical) TDGs for Fair Coin

Events are Heads, Tails: $\Omega = \{Heads, Tails\}$.
A gamble g is an element of $\mathbb{R}^2$ $g = [g_1, g_2]$.
If Alice accepts g then:

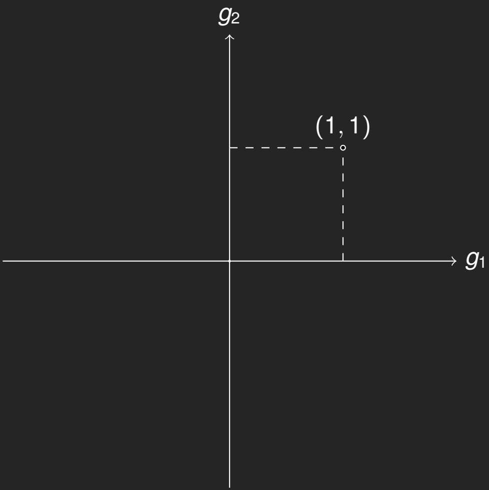
- she commits herself to receive/pay g_1 if Heads;
- she commits herself to receive/pay g_2 if Tails.

We ask Alice to state whether a certain gamble is **desirable** for her, meaning that she would commit herself to accept whatever reward or loss it will eventually lead to.

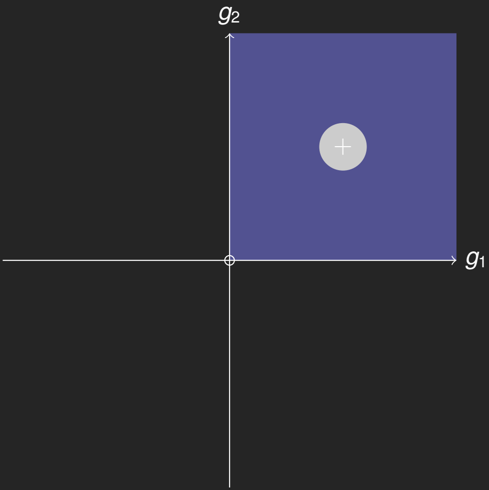
What's desirable for Alice?



What's desirable for Alice?

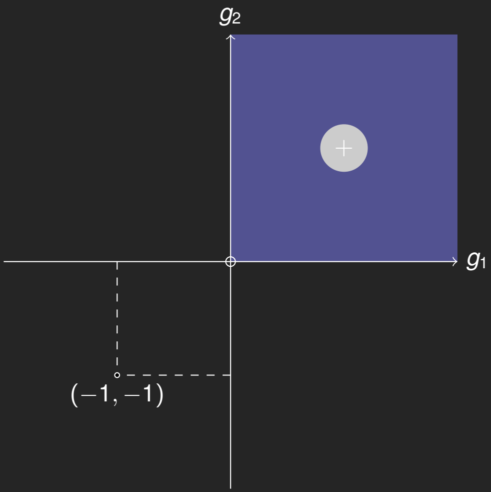


What’s desirable for Alice?

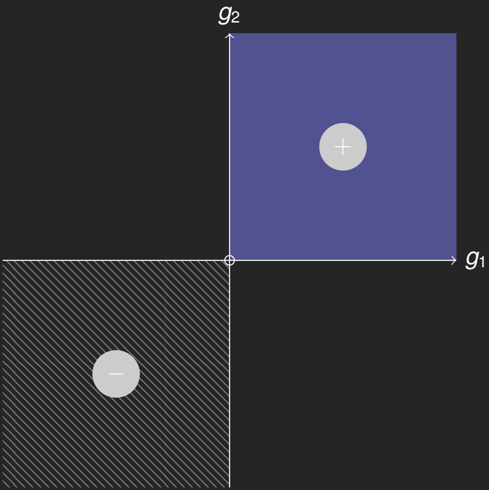


Any gamble $g \neq 0$ such that $g(\omega) \geq 0$ for each $\omega \in \Omega$ must be desirable for Alice.

What’s desirable for Alice?

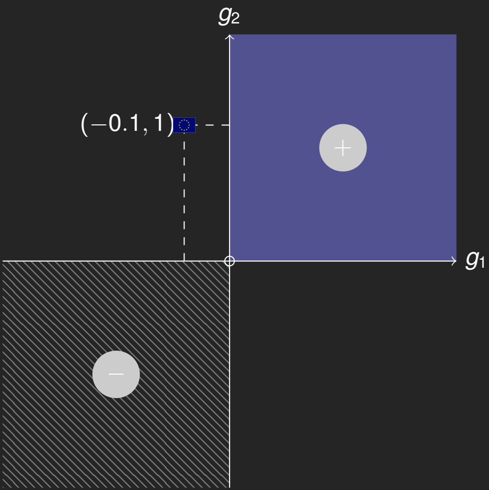


What's desirable for Alice?

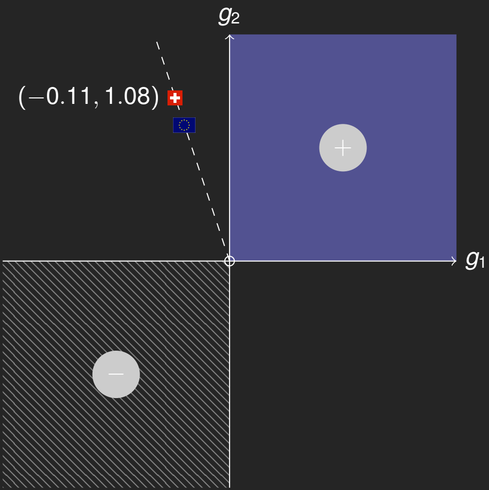


Any gamble g such that $g(\omega) \leq 0$ for each $\omega \in \Omega$ must not be desirable for Alice.

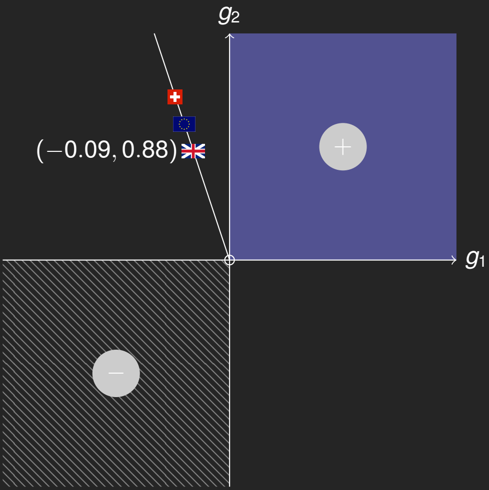
What’s desirable for Alice?



What's desirable for Alice?

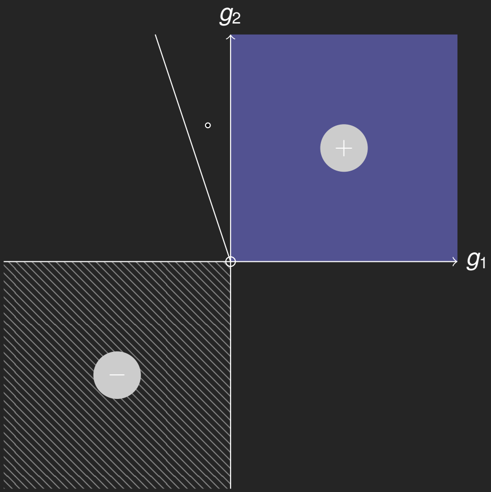


What’s desirable for Alice?

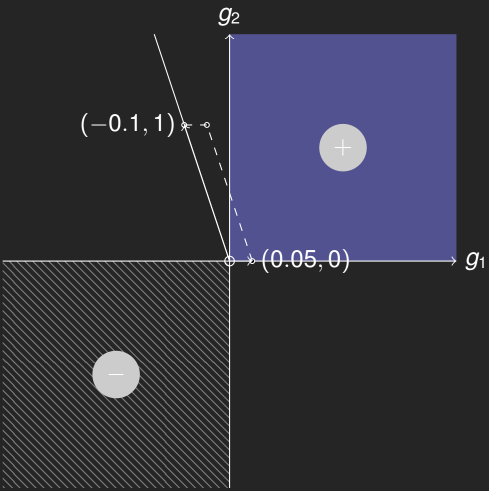


If Alice finds g to be desirable ($g \in \mathcal{K}$), then also λg must be desirable for any $0 < \lambda \in \mathbb{R}$.

What’s desirable for Alice?

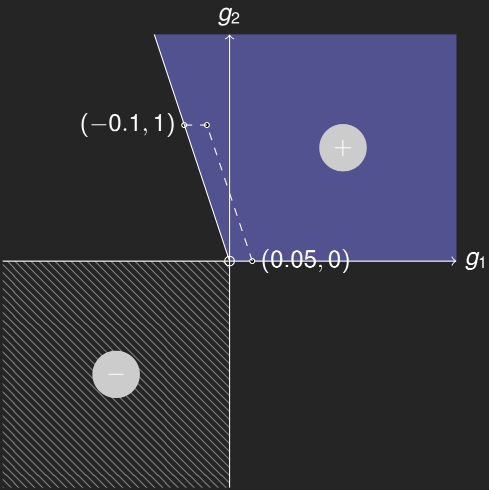


What’s desirable for Alice?



If Alice finds g_1, g_2 desirable ($g_1, g_2 \in \mathcal{K}$), then she also must accept $g_1 + g_2$.

What's desirable for Alice?



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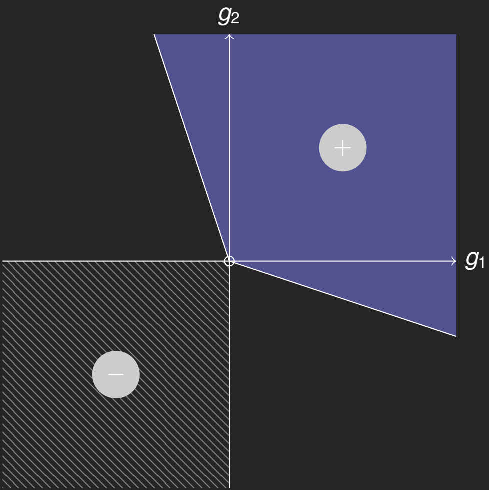

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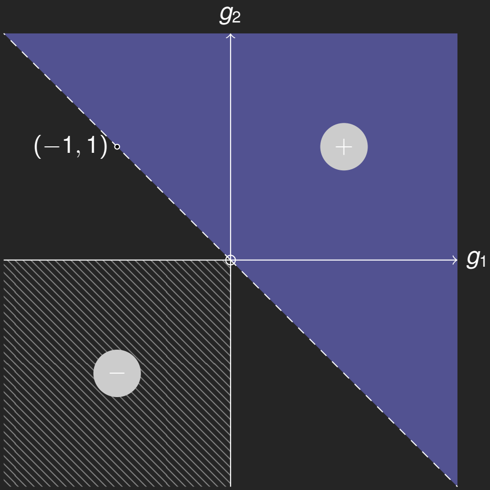
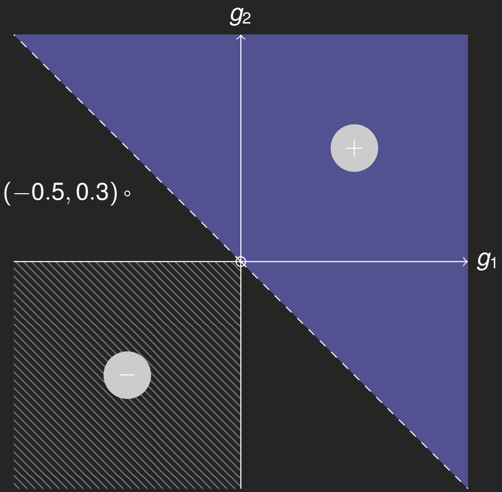
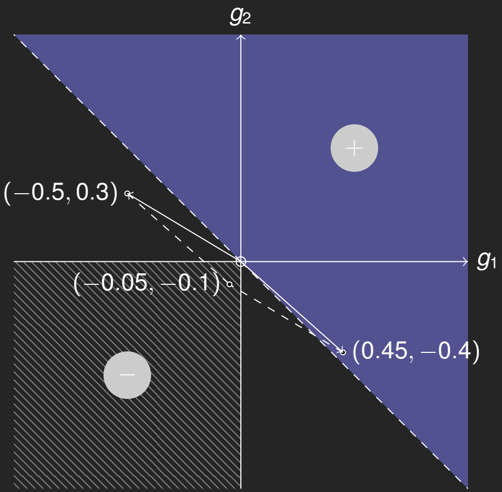


If $g \in \mathcal{K}$ then either $g \gneq 0$ or $g - \delta \in \mathcal{K}$ for some $0 < \delta \in \mathbb{R}^n$.

Sure Loss (Dutch Book)



Sure Loss (Dutch Book)



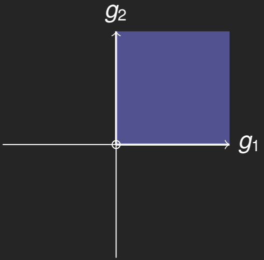
Summing up: (Classical) TDGs

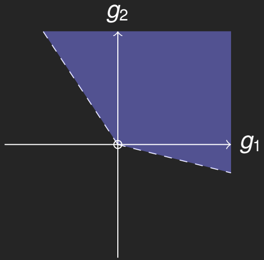
Definition 1 (Coherence)

The set $\mathcal{K}$ of Alice's desirable gambles is said to be **coherent** (**rational, consistent**) when it satisfies a few simple rationality criteria:

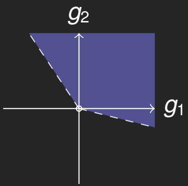
1. Accepting Positive gambles;
2. Avoiding Negative gambles;
3. Positive scaling ("change of currency");
4. Additivity ("parallelogram rule");
5. *Openness*.

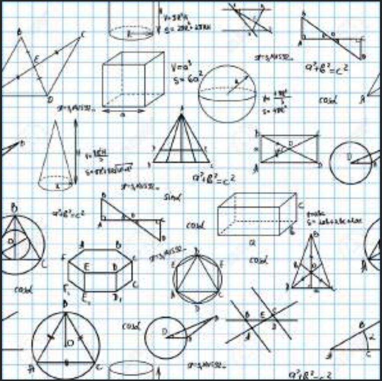
State of full or partial ignorance





Geometric properties ?





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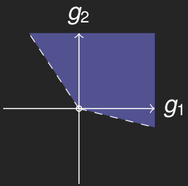
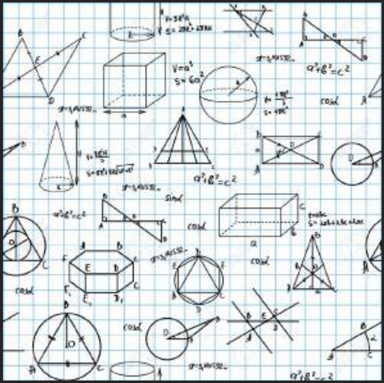
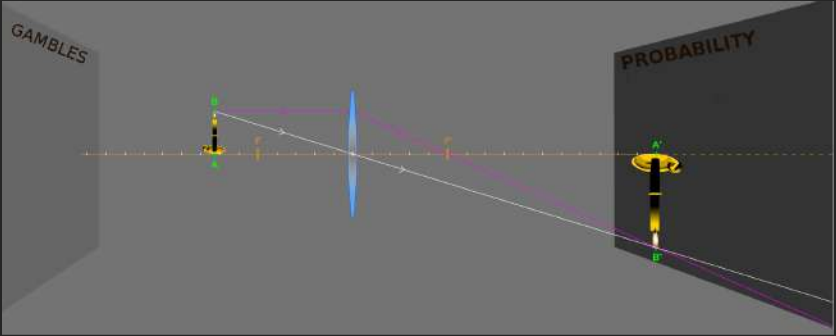




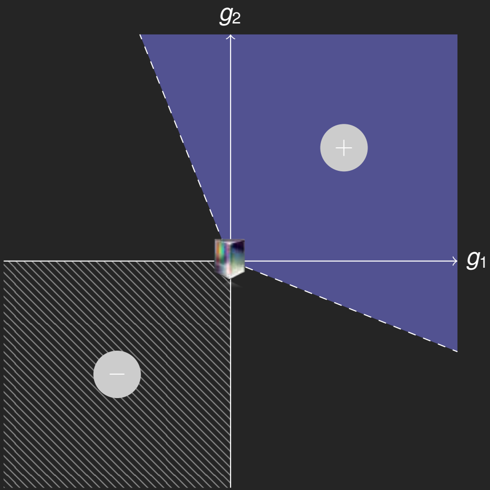
Alice through the looking glass

Probability does not exist.
B. de Finetti

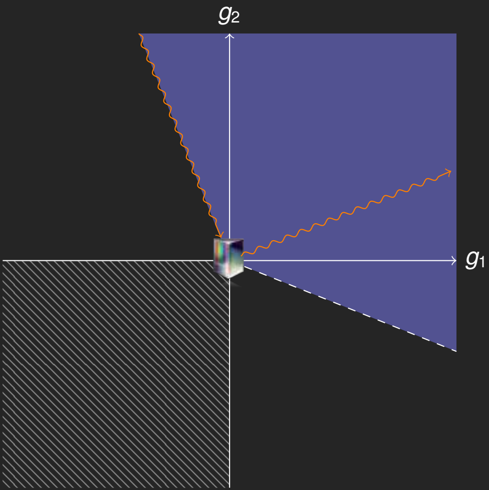


Probability rules can be derived from desirability via **Duality**.

Optics of Desirability (Duality)

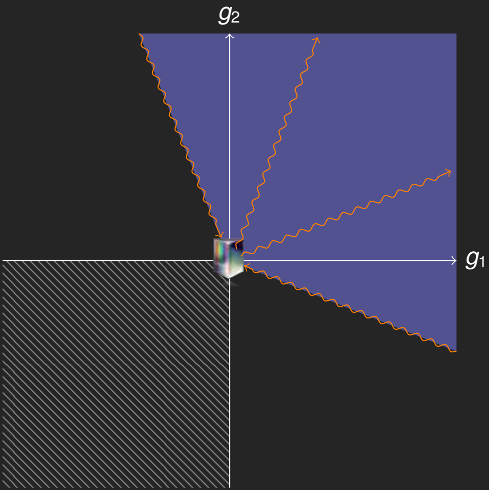


Optics of desirability



$g \perp t \Leftrightarrow g \cdot t = 0$

Optics of desirability



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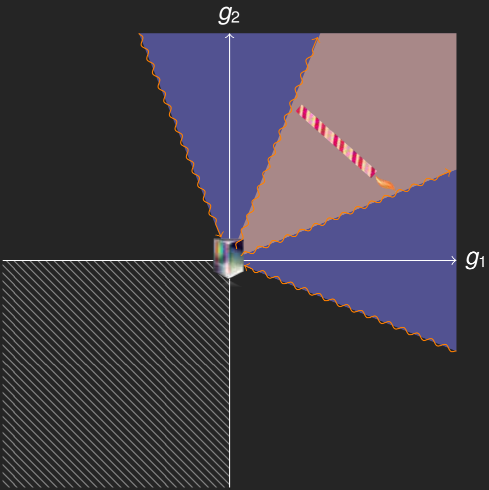
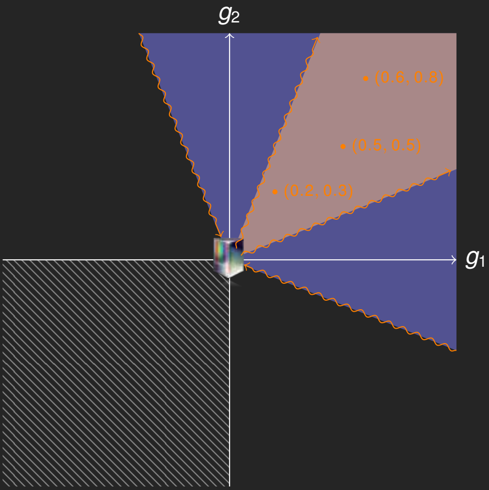
Optics of desirability



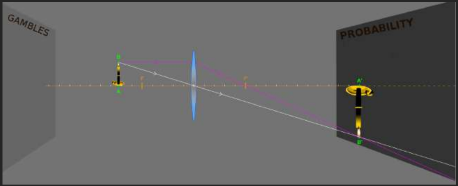
$\mathcal{K}^\bullet = \{t \in \mathbb{R}^n \mid \; g \cdot t \geq 0 \; \forall g \in \mathcal{K}\}$

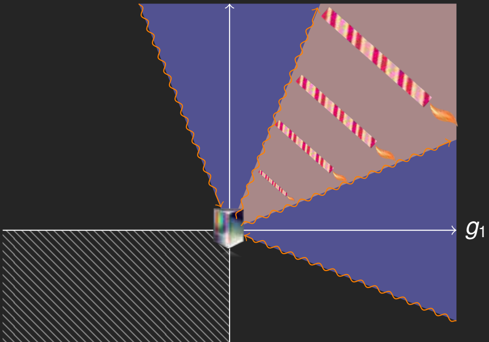
Polar cone



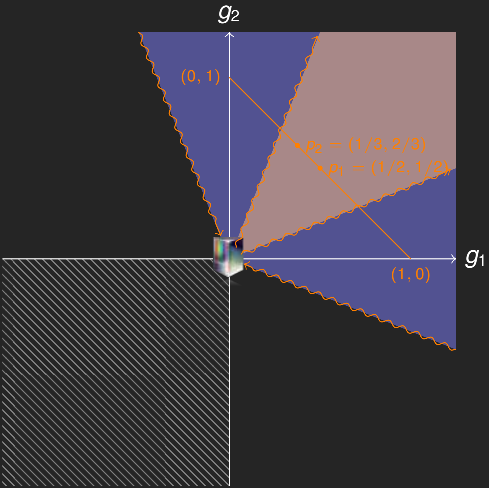
$\mathcal{K}^\bullet = \{t \in \mathbb{R}^n \mid t \geq 0,\ g \cdot t \geq 0\ \forall g \in \mathcal{K}\}$

Preserving the scale



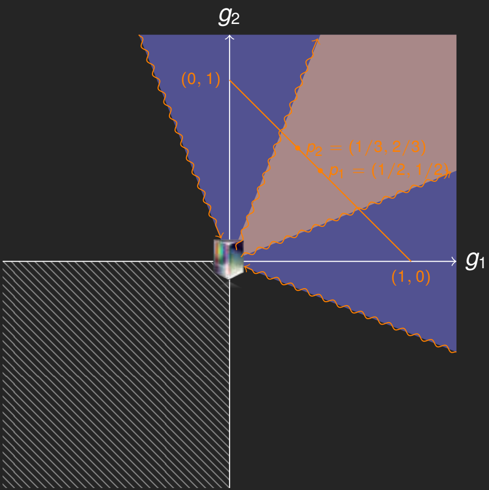


Preserving the scale



$$\mathcal{K}^\bullet = \{t \in \mathbb{R}^n \mid t \geq 0,\ \ 1 \cdot t = 1,\ \ g \cdot t \geq 0\ \forall g \in \mathcal{K}\}$$
$$= \{p \in \mathcal{P} \mid\ g \cdot p \geq 0\ \forall g \in \mathcal{K}\}$$

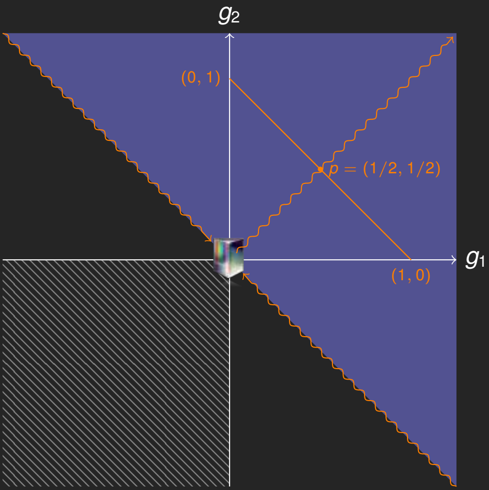
Preserving the scale



Imprecise Probability!

$$\mathcal{K}^\bullet = \{t \in \mathbb{R}^n \mid t \geq 0,\ \ 1 \cdot t = 1,\ \ g \cdot t \geq 0\ \forall g \in \mathcal{K}\}$$
$$= \{p \in \mathcal{P} \mid \ g \cdot p \geq 0\ \forall g \in \mathcal{K}\}$$

Fair Coin



$\mathcal{K}^\bullet = \{p = (1/2, 1/2)\}$

From (Classical) TDGs to probability axioms

From a few simple rationality criteria:

1. Accepting Positive gambles;
2. Avoiding Negative gambles;
3. Positive scaling ("change of currency");
4. Additivity ("parallelogram rule");
5. *Openness*.

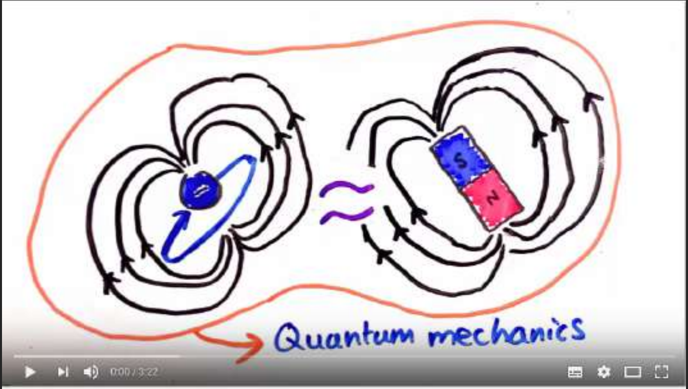
We can derive the rule of probabilities:

1. Probability is a number between 0 and 1.
2. Probability of the certain event is 1.
3. Probability of the event "A or B" is $P(A \vee B) = P(A) + P(B)$ (if the events are mutually exclusive);
4. the conditional probability of B given the event A is defined by Bayes' rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \text{ with } P(A) > 0.$$

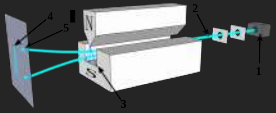
Quantum Mechanics

Stern-Gerlarch experiment



STERN-GERLACH & Electron Spin - by Looking Glass Universe BETTER THAN PHYSICS GIRL

Feynman's notation



$$\underset{Z}{\left\{ \begin{array}{l} +\vert \\ -\vert \end{array} \right\}} \qquad \underset{Z}{\left\{ \begin{array}{l} + \\ -\vert \end{array} \right\}} \qquad \underset{Z}{\left\{ \begin{array}{l} +\vert \\ - \end{array} \right\}} \qquad \underset{Z}{\left\{ \begin{array}{l} + \\ - \end{array} \right\}}$$

We can now compose SG apparatus in series:

$$\underset{S}{\left\{ \begin{array}{l} + \\ -\vert \end{array} \right\}} \xrightarrow{N} \underset{T}{\left\{ \begin{array}{l} +\vert \\ - \end{array} \right\}} \xrightarrow{\alpha N} \underset{R}{\left\{ \begin{array}{l} +\vert \\ - \end{array} \right\}} \xrightarrow{\beta\alpha N}$$

Feynman's notation

Odd example:

$$\underset{Z}{\left\{\begin{array}{c} + \\ -| \end{array}\right\}} \xrightarrow{N} \underset{X}{\left\{\begin{array}{c} + \\ -| \end{array}\right\}} \xrightarrow{\frac{1}{2}N} \underset{Z}{\left\{\begin{array}{c} +| \\ - \end{array}\right\}} \xrightarrow{\frac{1}{4}N}$$

$$\underset{Z}{\left\{\begin{array}{c} + \\ -| \end{array}\right\}} \xrightarrow{N} \underset{X}{\left\{\begin{array}{c} +| \\ - \end{array}\right\}} \xrightarrow{\frac{1}{2}N} \underset{Z}{\left\{\begin{array}{c} +| \\ - \end{array}\right\}} \xrightarrow{\frac{1}{4}N}$$

Feynman's notation

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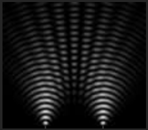
$$\underset{Z}{\left\{\begin{array}{l} + \\ -| \end{array}\right\}} \xrightarrow{N} \underset{X}{\left\{\begin{array}{l} + \\ - \end{array}\right\}} \xrightarrow{N} \underset{Z}{\left\{\begin{array}{l} +| \\ - \end{array}\right\}} \xrightarrow{0}$$



Playing with Mat

*Every **propositional** system can be embedded into a **projective geometry** in some **linear vector space** with coefficients from a **field**.*

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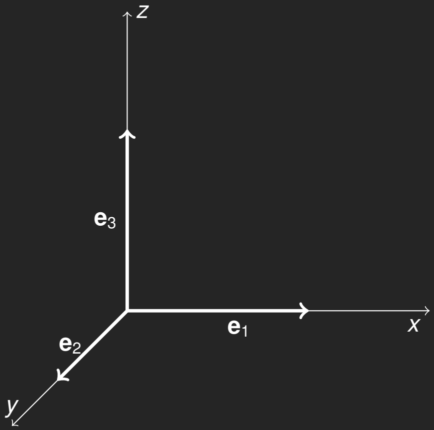

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

$$\begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix}$$

How do I say: “it comes out Heads and Alice gets g_1”

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

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How do I say: "it comes out Heads and Alice gets g_1"

$$\mathbf{e}_1\mathbf{e}_1^T \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \mathbf{e}_1\mathbf{e}_1^T = g_1 \quad \mathbf{e}_1\mathbf{e}_1^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = g_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Playing with Mat

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$$\mathbf{e}_1\mathbf{e}_1^T \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \mathbf{e}_1\mathbf{e}_1^T = g_1 \mathbf{e}_1\mathbf{e}_1^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = g_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{gI_H}$$

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

$$\begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix}$$

"it comes out either Heads or Tails and Alice gets either g_1 or g_2"

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

$$\begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix}$$

“it comes out either Heads or Tails and Alice gets either g_1 or g_2”

$$\mathbf{e}_1\mathbf{e}_1^T \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} \mathbf{e}_2\mathbf{e}_2^T = \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & 0 \end{bmatrix}$$

Why the canonical basis?

*Every **propositional** system can be embedded into a **projective geometry** in some **linear vector space** with coefficients from a **field**.*

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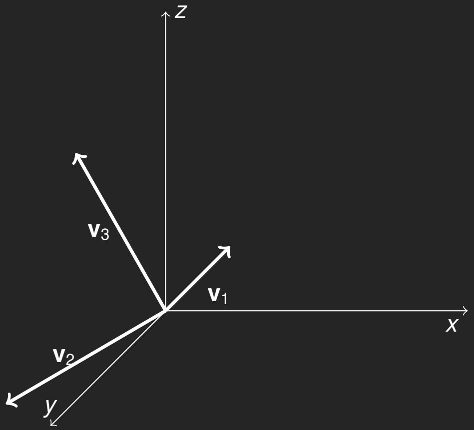

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

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$$\mathbf{v}_i = R\mathbf{e}_i$$

How do I say: “it comes out Heads and Alice gets g_1”

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

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$$\mathbf{v}_i = R\mathbf{e}_i$$

How do I say: "it comes out Heads and Alice gets g_1"

$$\mathbf{v}_1\mathbf{v}_1^T \quad \underbrace{(R^T)^{-1} \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} (R)^{-1}}_{} \quad \mathbf{v}_1\mathbf{v}_1^T \quad = \quad g_1 \quad \mathbf{v}_1\mathbf{v}_1^T$$

$$\mathbf{v}_1\mathbf{v}_1^T \qquad G \qquad \mathbf{v}_1\mathbf{v}_1^T \quad = \quad g_1 \quad \mathbf{v}_1\mathbf{v}_1^T$$

Girolamo Cardano Riddle

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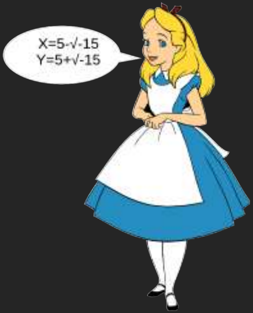

$$\iota = \sqrt{-1}$$

Let's change the field then

*Every **propositional** system can be embedded into a **projective geometry** in some **linear vector space** with coefficients from a **field**.*

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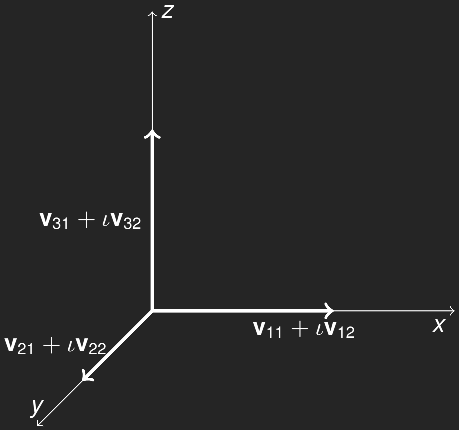

Playing with Mat

$$\begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

$$\begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix}$$

How do I say: “it comes out Heads and Alice gets g_1”

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How do I say: “it comes out Heads and Alice gets g_1”

$$\mathbf{v}_1\mathbf{v}_1^\dagger \quad \underbrace{(R^\dagger)^{-1} \begin{bmatrix} g_1 & & \\ & g_2 & \\ & & g_3 \end{bmatrix} (R)^{-1}} \quad \mathbf{v}_1\mathbf{v}_1^\dagger \quad = \quad g_1 \quad \mathbf{v}_1\mathbf{v}_1^\dagger$$

$$\mathbf{v}_1\mathbf{v}_1^\dagger \quad G \quad \mathbf{v}_1\mathbf{v}_1^\dagger \quad = \quad g_1 \quad \mathbf{v}_1\mathbf{v}_1^\dagger$$

$$\Pi_1 \quad G \quad \Pi_1 \quad = \quad g_1 \quad \Pi_1$$

Gambles for a Quantum coin (Electron Spin)

G is a Hermitian matrix (complex square matrix that is equal to its own conjugate transpose).

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 1 + \iota 2 \\ 1 - \iota 2 & -1 \end{bmatrix}$$

Protocol

- A *n*-dimensional quantum system is prepared by the bookmaker in some quantum state. Alice has her personal knowledge about the experiment (possibly no knowledge at all).

$$\underset{S}{\left\{\begin{array}{c} + \\ -| \end{array}\right\}} \xrightarrow{N} \underset{T}{\left\{\begin{array}{c} +| \\ - \end{array}\right\}} \xrightarrow{\alpha N} \underset{R}{\left\{\begin{array}{c} +| \\ - \end{array}\right\}} \xrightarrow{\beta\alpha N}$$

[1] We mean the eigenvectors of the density matrix of the quantum system.

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- The bookie announces that he will measure the quantum system along its n orthogonal directions, that is $\Omega = \{\omega_1, \dots, \omega_n\}$, with ω_i denoting the elementary event "detection along i". Mathematically, it means that the quantum system is measured along its eigenvectors,[1] i.e., the projectors $\Pi^* = \{\Pi_1^*, \dots, \Pi_n^*\}$.

[1] We mean the eigenvectors of the density matrix of the quantum system.

Protocol

- Before the experiment, Alice declares the set of gambles she is willing to accept. Mathematically, a gamble G on this experiment is a Hermitian matrix, i.e., $G \in \mathbb{C}_h^{n \times n}$. We will denote the set of gambles Alice is willing to accept by $\mathcal{K} \subseteq \mathbb{C}_h^{n \times n}$.

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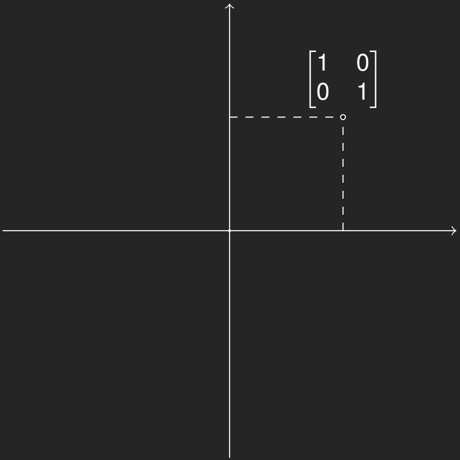
- By accepting a gamble G, Alice commits herself to receive $\gamma_i \in \mathbb{R}$ euros if the outcome of the experiment eventually happens to be ω_i. The value γ_i is defined from G and Π^* as follows:

$$\Pi_i^* \quad G \quad \Pi_i^* \quad = \quad g_i \quad \Pi_i^*$$

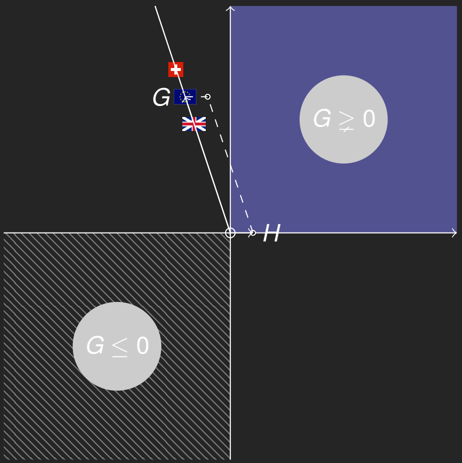
It is a real number since G is Hermitian.

What’s desirable for Alice? (pictorial)



$$\Pi_i^* \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Pi_i^* = 1 \quad \Pi_i^*$$

What's desirable for Alice? (pictorial)



$\Pi_i^* G \Pi_i^* = \gamma_i \Pi_i^*$ for $i = 1, \ldots, n$.

Summing up: Quantum TDGs

Definition 2 (Coherence)

The set $\mathcal{K}$ of Alice's desirable Gambles is said to be **coherent** (**rational, consistent**) when it satisfies a few simple rationality criteria:

1. Accepting Positive *G*ambles;
2. Avoiding Negative *G*ambles;
3. Positive scaling ("change of currency");
4. Additivity ("parallelogram rule");
5. *Openness*.

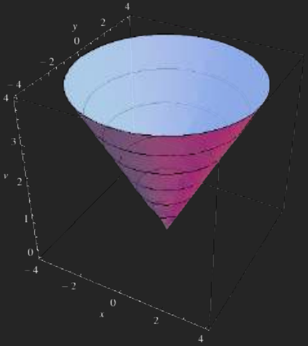
I use the word Gambles (capital G) for matrix gambles.

Geometric Properties

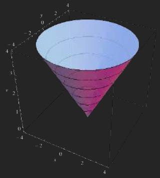
By exploiting Pauli decomposition, any 2D Hermitian matrices can be written as:

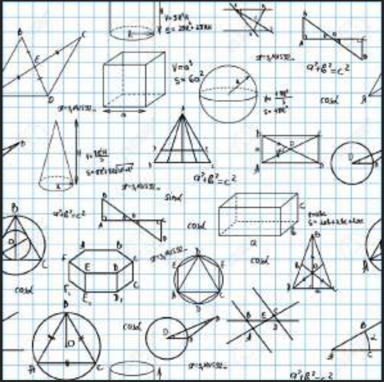
$$G = \begin{bmatrix} v+z & x-\iota y \\ x+\iota y & v-z \end{bmatrix} = vI + x\sigma_x + y\sigma_y + z\sigma_z,$$

3D projection of the cone of all positive semi-definite matrices.



Geometric properties ?





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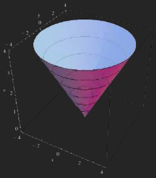
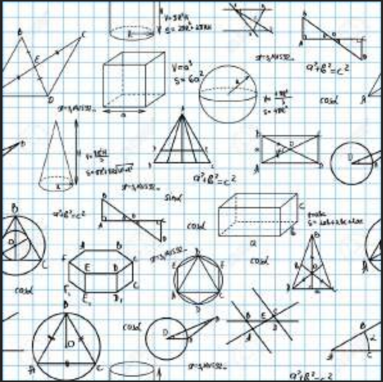
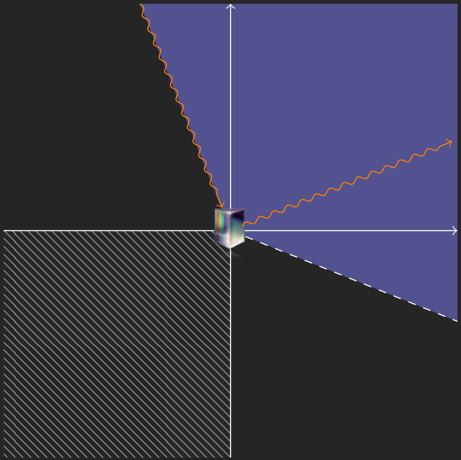


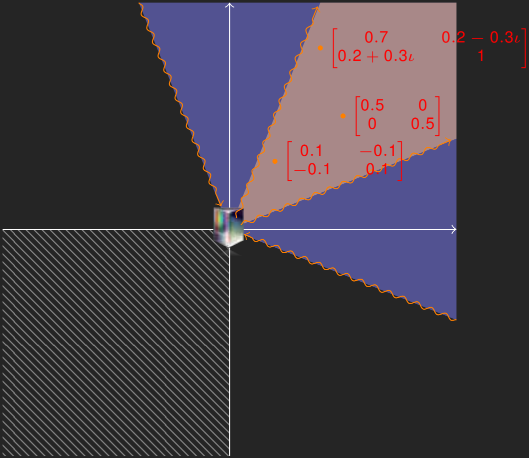


Optics of Quantum Desirability (pictorial)



$G \perp R \Leftrightarrow G \cdot R = Tr(G^{\dagger} R) = 0$

Optics of Quantum Desirability



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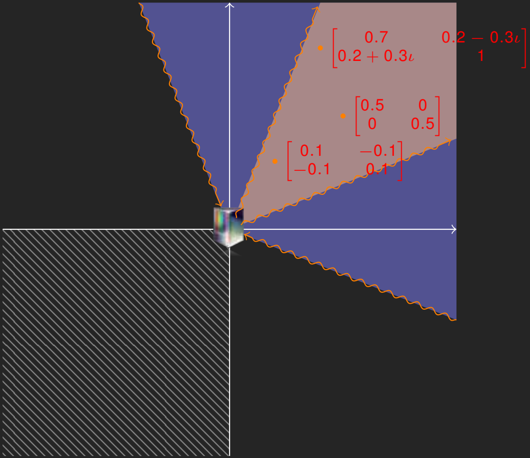

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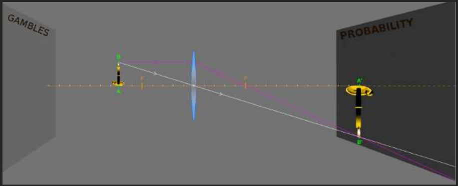
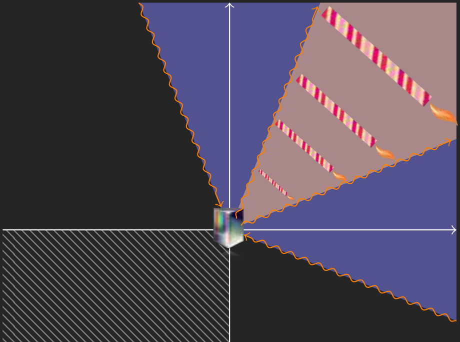
Preserving the scale





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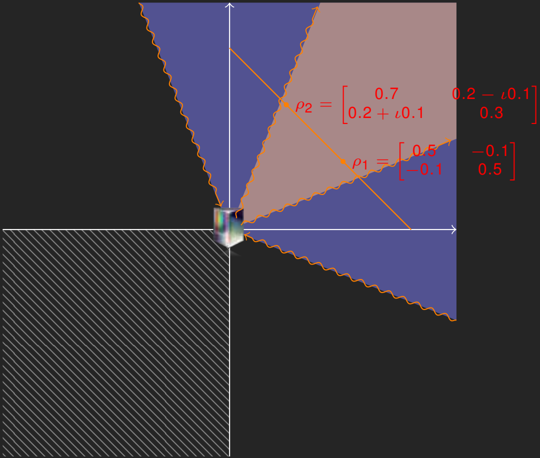

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Duality of coherence

- The dual of Alice's coherent set of strictly desirable gambles is the set

$$\mathcal{M} = \{\rho \in \mathcal{D}_h^{n\times n} \mid \rho \geq 0,\ Tr(\rho) = 1,\ \ G \cdot \rho \geq 0\ \forall G \in \mathcal{K}\},$$

  that includes all positive operators with trace one (i.e., density matrices), that are compatible with Alice's beliefs about the quantum system (expressed in terms of desirable gambles).

This is exactly the first axiom of QM,

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> *Associated to any isolated physical system is a complex Hilbert space known as the state space of the system. The system is completely described by its density operator, which is a positive operator ρ with trace one, acting on the state space of the system.*

expressed in a completely subjective way.

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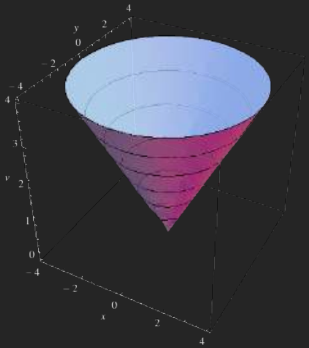
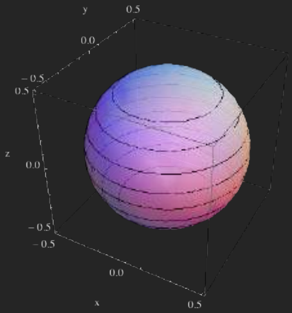
 $\oplus$  $=$ *QM*

State of full ignorance about a QuBit experiment



3D projection of the cone of all positive semi-definite matrices.

Dual: Bloch Sphere



$$\mathcal{M} \quad = \quad \mathcal{D}_h^{n\times n} \rightarrow \text{ All density matrices}$$

Maximal knowledge about a QuBit experiment

Alice's SDG $\mathcal{K}$ this time coincides with

$$\mathcal{K} = \{G \in \mathbb{C}_h^{n\times n} \mid G \gneq 0\} \cup \{G \in \mathbb{C}_h^{n\times n} \mid Tr(G^\dagger D) > 0\},$$

$$D = \frac{1}{2}\begin{bmatrix} 1 & -\iota \\ \iota & 1 \end{bmatrix},$$

where ι denotes the imaginary unit.

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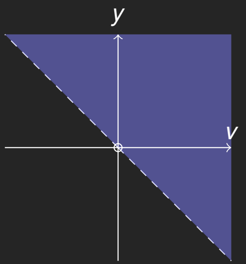
$$G = \begin{bmatrix} v + z & x - \iota y \\ x + \iota y & v - z \end{bmatrix} = vI + x\sigma_x + y\sigma_y + z\sigma_z,$$

we obtain $Tr(G^\dagger D) = v + y > 0$



Maximal knowledge about a QuBit experiment

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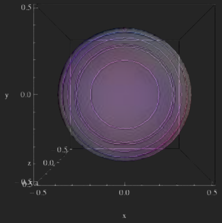
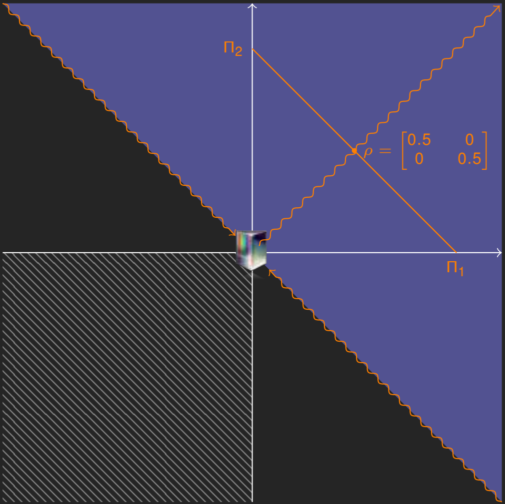


$$\mathcal{M} \quad = \quad \{\rho = D\}$$

Classical probability is “included” in QM

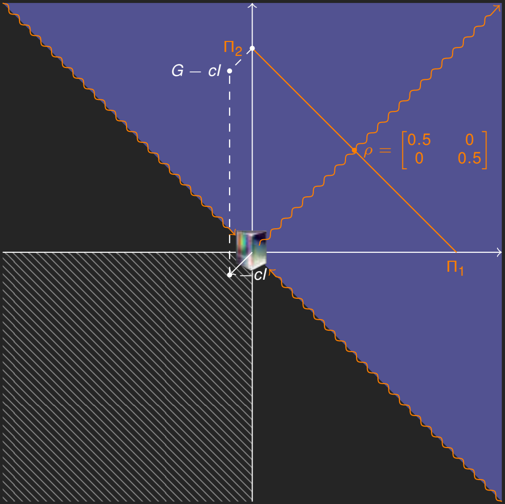
$$\Pi_1 = \mathbf{e}_1\mathbf{e}_1^\dagger, \quad \Pi_2 = \mathbf{e}_2\mathbf{e}_2^\dagger$$



Fair Price

What is Alice's fair price for the gamble $G = \Pi_2 = \mathbf{e}_2\mathbf{e}_2^\dagger$?

$$\max c : \quad G - cI \in \mathcal{K}$$



$$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$cI = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$G - cI = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.9 \end{bmatrix}$$

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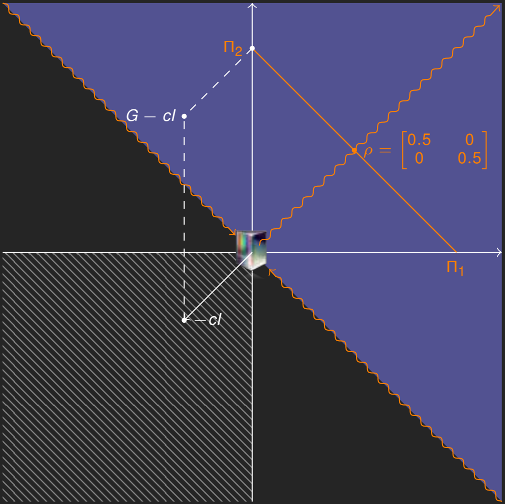
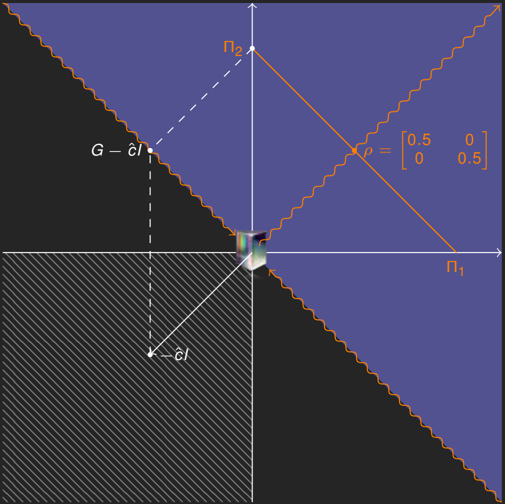


$$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$cI = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$G - cI = \begin{bmatrix} -0.4 & 0 \\ 0 & 0.6 \end{bmatrix}$$

Fair Price

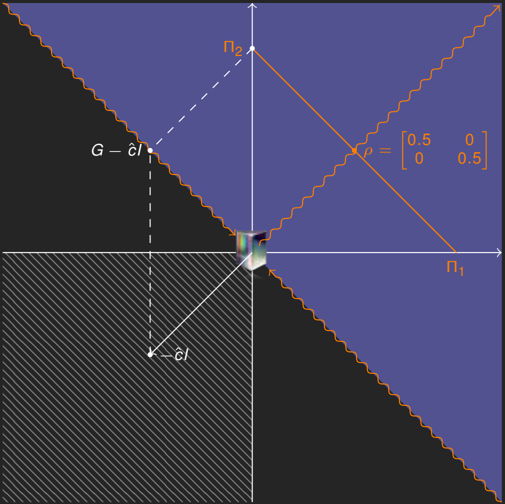


$$G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{c}I = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$G - \hat{c}I = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Fair Price



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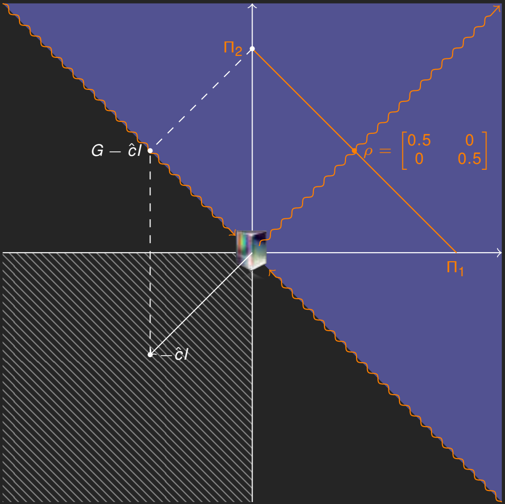
$\hat{c}$ is Alice's fair price for the gamble G

Fair Price through Duality



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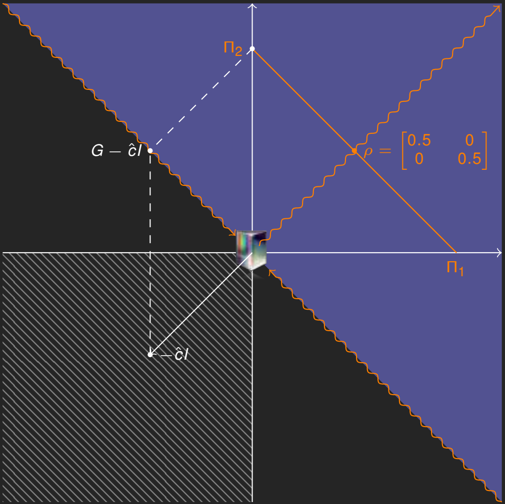

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Duality: we can show that

$\hat{c} = Tr(G\rho) = Tr(\Pi_2\rho) = \rho_{22}$ so $\hat{c}$ is Alice's probability for the event Π_2 (Tails)

In QM: fair Price of an Event (Projector)

We have learned that

$$p_1 = Tr(\Pi_1 \rho), \;\; p_2 = Tr(\Pi_2 \rho), \ldots, p_n = Tr(\Pi_n \rho)$$

$$1 = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} Tr(\Pi_i \rho)$$

We have derived

Born's rule

as subjective fair price of a gamble.

In case of non-maximal cones we obtain lower and upper probabilities.

Quantum Coin

Assume that Alice's SDG is

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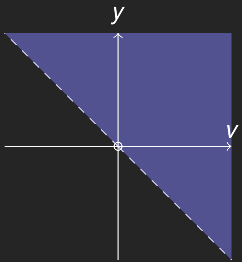

Alice's belief on the result of a SG experiment

By duality we have seen that

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & -\iota \\ \iota & 1 \end{bmatrix}$$

Assume we want to know what are Alice's probabilities of observing Z_+ and Z_-

$$\left\{ \begin{array}{c} +| \\ -| \end{array} \right\}$$

$$Z$$

$Z_+ = \mathbf{e}_1\mathbf{e}_1^\dagger$ and $Z_- = \mathbf{e}_2\mathbf{e}_2^\dagger$

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$$\underset{Z}{\left\{ \begin{array}{c} +| \\ -| \end{array} \right\}}$$

$Z_+ = \mathbf{e}_1\mathbf{e}_1^\dagger$ and $Z_- = \mathbf{e}_2\mathbf{e}_2^\dagger$

$$p_1 = Tr(\Pi_{Z_+}\rho) = \frac{1}{2}, \;\; p_2 = Tr(\Pi_{Z_-}\rho) = \frac{1}{2}$$

So Alice believes that the probability of observing Z_+ and Z_- is $\frac{1}{2}$.

Alice's belief on the result of a SG experiment

By duality we have seen that

$$\rho = \frac{1}{2}\left[\begin{array}{cc} 1 & -\iota \\ \iota & 1 \end{array}\right]$$

Assume we want to know what are the probabilities of Alice's probabilities of observing Y_+ and Y_-

$$\left\{\begin{array}{c} +| \\ -| \end{array}\right\}$$

$$Y$$

$$\Pi_{Y_+} = \begin{bmatrix} \frac{1}{2} & -\iota\frac{1}{2} \\ \iota\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \Pi_{Y_-} = \begin{bmatrix} \frac{1}{2} & \iota\frac{1}{2} \\ -\iota\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

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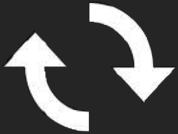
$$p_1 = Tr(\Pi_{Z_+}\rho) = 1, \ \ p_2 = Tr(\Pi_{Z_-}\rho) = 0$$

So Alice believes that the probability of observing Y_+ is 1.

Last missing brick

The theory of DG is subjective (epistemic) but **Quantum Experiments** are real. Different subjects (Alice, Bob, Charlie...) must be able to reach the same conclusion conditional on some evidence.



We need a rule for **updating** a SDG based on new **evidence** (from quantum experiments).

Coherent Updating

Assume that Alice considers an event "indicated" by a certain projector Π_i in $\Pi = \{\Pi_i\}_{i=1}^n$.

Alice can focus on gambles that are contingent on the event Π_i:

> *these are gambles such that "outside"* Π_i *no utile is received or due – status quo is maintained*

Mathematically, these gambles are of the form

$$G = \begin{cases} H & \text{if } \Pi_i \text{ occurs,} \\ 0 & \text{if } \Pi_j \text{ occurs, with } j \neq i. \end{cases}$$

or, equivalently,

$$G = \alpha \Pi_i$$

for some $\alpha \in \mathbb{R}$.

Coherent Updating

Definition 3

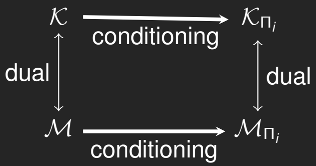
Let $\mathcal{K}$ be an SDG, the set obtained as

$$\mathcal{K}_{\Pi_i} = \left\{ G \in \mathbb{C}_h^{n \times n} \mid G \gneq 0 \textit{ or } \Pi_i G \Pi_i \in \mathcal{K} \right\}$$

is called the **set of desirable gambles conditional** on Π_i.

We can also compute the dual of $\mathcal{K}_{\Pi_i}$, i.e., $\mathcal{M}_{\Pi_i}$ – we call it a **conditional quantum credal set**.

Does this digram commute?



Coherent Updating

Definition 3

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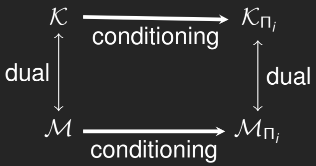

By duality, we can show that this digram commutes when $Tr(\Pi_i \rho \Pi_i) = Tr(\Pi_i \rho) > 0$.

Subjective formulation of the second axiom of QM

Given a quantum credal set $\mathcal{M}$, the corresponding quantum credal set conditional on Π_i is obtained as

$$\mathcal{M}_{\Pi_i} = \left\{ \frac{\Pi_i \rho \Pi_i}{Tr(\Pi_i \rho \Pi_i)} \middle| \rho \in \mathcal{M} \right\},$$

provided that $Tr(\Pi_i \rho \Pi_i) > 0$ for every $\rho \in \mathcal{M}$.

This rule is called in QM

Luders' Rule

or the **collapse of the wave function**, because after the measurement the new density matrix is equal to Π_i with certainty.

What actually is this rule?

Let us consider the case $\rho = \mathit{diag}(0.5, 0.5)$, i.e., she believes that the coin is fair and

$$\Pi_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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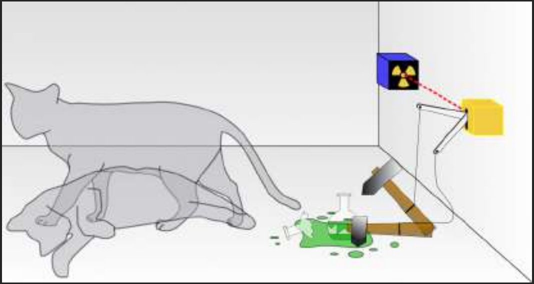
Under the assumption that "the coin has landed head up", Alice's knowledge about the coin experiment "has collapsed" to $p = [1, 0]$ – she knows that the result of the experiment is Head.

This "solves" the cat dilemma



Alice may believe that the cat is alive or dead (in her imagination), when she opens the box she is simply updating her beliefs.

Conclusions: QM as desirability

Theory of desirability	QM
Rationality	Density Matrix (1st axiom)
Conditioning	Measurement (2d axiom)
Temporal coherence	Time Evolution (3d axiom)
Epistemic Independence	Separable States (4th axiom)

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