# Efficient Computation of Upper Probabilities of Failure Using Monte Carlo Simulation and Reweighting Techniques

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## Poster Abstract

Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of random variables. The upper probability $\overline{p}_f$ of failure is the solution of the optimization problem

$$\overline{p}_f = \max_{\lambda\in\Lambda} \int_D \mathbb{1}_{g(x)\le 0}\; f_{X_\lambda}(x)\,\mathrm{d}x$$

where $f_{X_\lambda}$ is the corresponding probability density of the random variable $X_\lambda$, $g : D \to \mathbb{R}$ a limit state function and $\mathbb{1}$ the indicator function. A value $g(x) \le 0$ means failure of the underlying engineering structure and a value $g(x) > 0$ means that the engineering structure is safe. As an example, $f_{X_\lambda}$ is the density function of a Gaussian random variable $X_\lambda \sim \mathcal{N}(\mu(\lambda), \Sigma(\lambda))$ with expectation $\mu$ and covariance matrix $\Sigma$ parametrized by $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$.

The objective function $p(\lambda) = \int_D \mathbb{1}_{g(x)\le 0}\; f_{X_\lambda}(x)\,\mathrm{d}x$ of the above optimization problem can be approximated using Monte Carlo simulation together with reweighting or importance sampling techniques using only one single sample $x_1, \ldots, x_N$ distributed according to a "basic" random variable $X^0$ with $f_{X^0} > 0$:

$$\begin{aligned} p(\lambda) = \int\limits_D \mathbb{1}_{g(x)\le 0}\; f_{X_\lambda}(x)\,\mathrm{d}x &= \int\limits_D \mathbb{1}_{g(x)\le 0}\; \frac{f_{X_\lambda}(x)}{f_{X^0}(x)}\, f_{X^0}(x)\,\mathrm{d}x \\ &\approx \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_{g(x_k)\le 0}\; \frac{f_{X_\lambda}(x_k)}{f_{X^0}(x_k)} =: p^{X^0}_{x_1,\ldots,x_N}(\lambda), \end{aligned}$$

cf. Fetz and Oberguggenberger (2016). Then an approximation of the upper probability $\overline{p}_f$ of failure is obtained by $\overline{p}_f \approx \max_{\lambda\in\Lambda} p^{X^0}_{x_1,\ldots,x_N}(\lambda)$. This method needs only $N$ function evaluations $g(x_1), \ldots, g(x_N)$ of the limit state function which is an advantage in cases where the evaluation of $g$ is time consuming, e.g. finite element computations. Further, $p^{X^0}_{x_1,\ldots,x_N}$ depends continuously on $\lambda$ (if $f_{X_\lambda}$ is continuous, too) which makes maximization easier. We also note that we get different functions $p^{X^0}_{x_1,\ldots,x_N}$ for different samples $x_1, \ldots, x_N$ and basic random variables $X^0$.

The purpose of the poster presentation is to discuss and compare variants of the above approach such as iterating and strategies for choosing the random variable $X^0$, and to exemplify these methods by an engineering example.

## References

T. Fetz and M. Oberguggenberger. Imprecise random variables, random sets, and Monte Carlo simulation. *International Journal of Approximate Reasoning*, 78:252–264, 2016.