Learning Tractable Multidimensional Bayesian Network Classifiers

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Agenda

1. Background
2. Tractable MBCs
3. Experimental Results
4. Conclusions and Future Research

Background

Multidimensional Bayesian Network Classifiers

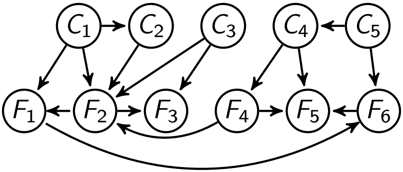


Figure 1: MBC structure

- **Multidimensional Bayesian network classifiers** (MBCs) (van der Gaag and de Waal, 2006) extend Bayesian network classifiers to the problem of multidimensional classification
- **Good performance** in multiple domains
- **Expressive** graphical representation

Multidimensional Bayesian Network Classifiers

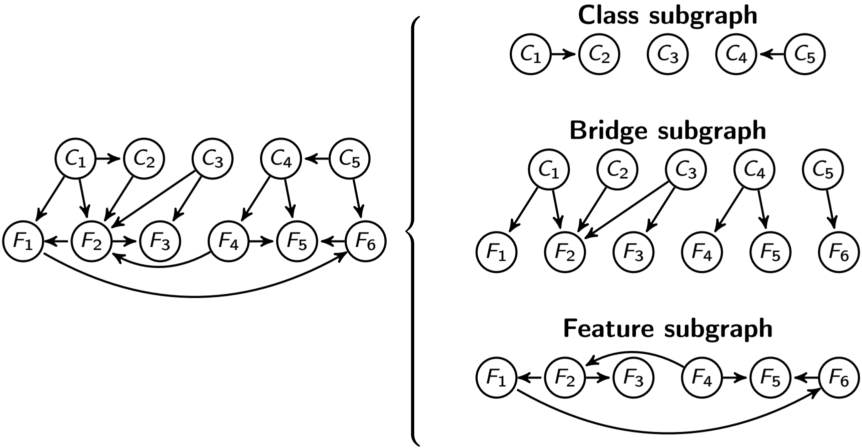


Figure 2: MBC subgraphs

Inference in MBCs I

- Performing multidimensional classification in an MBC is equivalent to **obtaining the MPE**

$$\max_{C_1,\ldots,C_5} \Pr(C_1,\ldots,C_5 | F_1,\ldots,F_6)$$

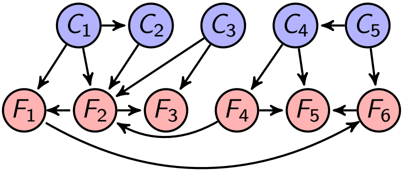


Figure 3: Multidimensional classification with an MBC

Inference in MBCs II

- If there are **unobserved feature variables**, multidimensional classification in MBCs is equivalent to **obtaining the MAP**. This can be intractable even if the treewidth is bounded (Park, 2002)

$$\max_{C_1,\dots,C_5} \Pr(C_1,\dots,C_5 | F_3,\dots,F_6)$$

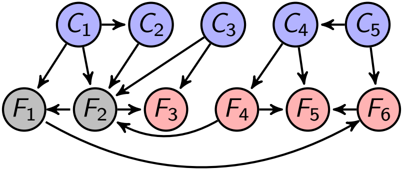


Figure 4: Multidimensional classification with an MBC

CB-Decomposable MBCs

The CB-decomposability of MBCs can **reduce the number of computations** required to perform multidimensional classification (Bielza et al., 2011)

$$\max_{\mathbf{c}\in\Omega\mathcal{C}} \Pr(\mathbf{c}|\mathbf{f}) \propto \prod_{i=1}^{r} \max_{\mathbf{c}_i\in\Omega\mathcal{C}_i} \prod_{C\in\mathcal{C}_i} \Pr(c|\mathbf{Pa}_{\mathcal{G}}(C)) \prod_{F\in\mathbf{Ch}_{\mathcal{G}}(\mathcal{C}_i)} \Pr(f|\mathbf{Pa}_{\mathcal{G}_B}(F), \mathbf{Pa}_{\mathcal{G}_F}(F))$$

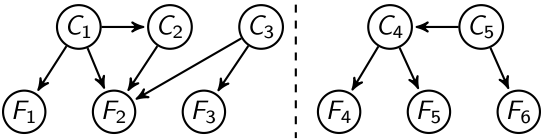


Figure 5: Connected components of an MBC

Inference Complexity in MBCs

- **MPE is generally NP-hard** (Kwisthout, 2011), but it can be computed in **polynomial time** in any BN if its **treewidth is bounded** (Sy, 1992)
- De Waal and van der Gaag (2007) demonstrated that treewidth($\mathcal{G}$) $\leq$ treewidth($\mathcal{G}_F$) $+ d$

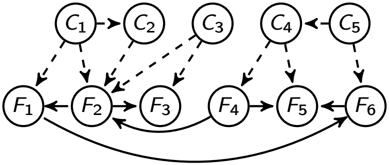


Figure 6: Class variables and feature subgraph of an MBC

Inference Complexity in MBCs

- Pastink and van der Gaag (2015) also suggested that the **complexity of an MBC** with an empty feature subgraph is given by the treewidth of the graph obtained by:
    1. **Moralizing** its structure
    2. **Removing all its feature** nodes from the moralized graph

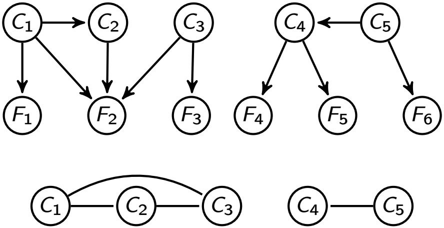


Figure 7: MBC structure (above) and moralized graph without feature nodes (below)

Learning MBCs

- Unconstrained complexity:
  - Tree–tree (van der Gaag and de Waal, 2006)
  - Polytree–polytree (de Waal and van der Gaag, 2007)
  - DAG–DAG (Bielza et al., 2011)

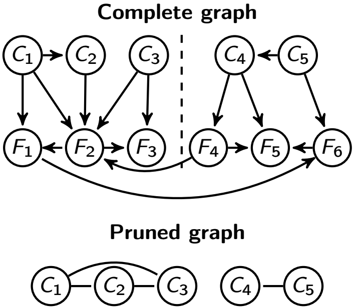
- Penalized complexity:
  - Polytree–empty (Corani et al., 2014)
  - Tree–empty (Pastink and van der Gaag, 2015)
  - CB–decomposable MBC (DAG–DAG) (Borchani et al., 2010)

Tractable MBCs

Motivation

- Previous research uses the **treewidth of the complete graph** to bound the complexity of multidimensional classification in MBCs
- We consider **query-dependent information**
- We provide a **new upper bound** for the complexity of multidimensional classification in MBCs
- Learn **DAG–DAG** MBCs

Tractable MBCs



	Before	After
ϕ_{C_1}	$\{C_1\}$	$\{C_1\}$
ϕ_{C_2}	$\{C_1, C_2\}$	$\{C_1, C_2\}$
ϕ_{C_3}	$\{C_3\}$	$\{C_3\}$
ϕ_{C_4}	$\{C_4, C_5\}$	$\{C_4, C_5\}$
ϕ_{C_5}	$\{C_5\}$	$\{C_5\}$
ϕ_{F_1}	$\{C_1, F_1, F_2\}$	$\{C_1\}$
ϕ_{F_2}	$\{C_1, C_2, C_3, F_2, F_4\}$	$\{C_1, C_2, C_3\}$
ϕ_{F_3}	$\{C_3, F_2, F_3\}$	$\{C_3\}$
ϕ_{F_4}	$\{C_4, F_4\}$	$\{C_4\}$
ϕ_{F_5}	$\{C_4, F_4, F_5, F_6\}$	$\{C_4\}$
ϕ_{F_6}	$\{C_5, F_1, F_6\}$	$\{C_5\}$

Figure 8: MBC structure and pruned graph (left), and domain of the potential of each node before and after they are updated with evidence $\mathbf{f} = (f_1, \ldots, f_6)$ (right)

Tractable MBCs

	This work	Pastink and van der Gaag (2015)
Feature subgraph	DAG	Empty
Bound	Pruned graph	Complete graph
Structure family	DAG–DAG	Tree–empty

Table 1: Differences between Pastink and van der Gaag (2015) and this work

Tractable MBCs

Theorem 1

Let $\mathcal{G} = (\mathcal{C} \cup \mathcal{F}, \mathcal{A}_C \cup \mathcal{A}_B \cup \mathcal{A}_F)$ be the structure of an MBC $\mathcal{B}$. If the ***treewidth of its pruned graph*** *$\mathcal{G}'$ and the number of parents of each node that belongs to $\mathcal{F}$ are* ***bounded***, *$\mathcal{B}$ can perform classification in* ***polynomial time***.

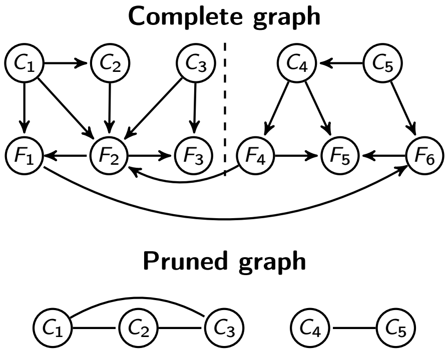


Figure 9: MBC structure and pruned graph

Tractable MBCs

Corollary 2

Let $\mathcal{G} = (\mathcal{C} \cup \mathcal{F}, \mathcal{A}_C \cup \mathcal{A}_B \cup \mathcal{A}_F)$ be the structure of an MBC $\mathcal{B}$. If the ***number of class variables*** *d and the number of parents of each node in $\mathcal{F}$ are* ***bounded****, $\mathcal{B}$ can perform classification in* ***polynomial time****.*

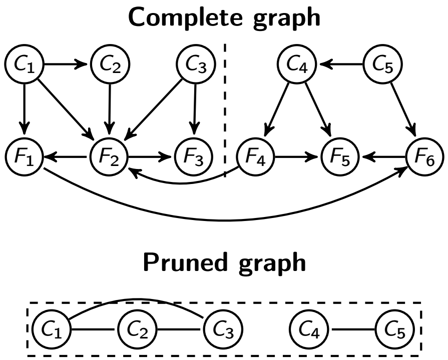


Figure 10: MBC structure and pruned graph

Tractable MBCs

Corollary 3

Let $\mathcal{G} = (\mathcal{C} \cup \mathcal{F}, \mathcal{A}_C \cup \mathcal{A}_B \cup \mathcal{A}_B)$ be the structure of a ***CB-decomposable*** *MBC $\mathcal{B}$. If the* ***number of class variables in each component*** *of $\mathcal{G}$ and the number of parents of each node in $\mathcal{F}$ are bounded, $\mathcal{B}$ can perform classification in* ***polynomial time****.*

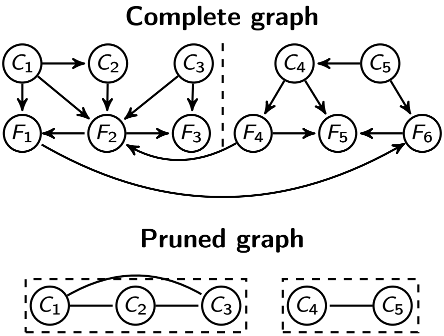


Figure 11: MBC structure and pruned graph

Learning Method

- We adapt **order-based search** (OBS) (Bouckaert, 1992) to learn tractable MBCs
- We learn CB-decomposable MBCs, **bounding** the maximum number of **class variables per component**
- Rejecting candidates that exceed the bound of class variables per component bounds the complexity of multidimensional classification
- To search in the space of orderings we use the procedure described by Teyssier and Koller (2005)

Experimental Results

Experimental Framework

- Learning Methods:
  - **CB-OBS**
  - Tree–tree (van der Gaag and de Waal, 2006)
  - Polytree–polytree (de Waal and van der Gaag, 2007)
  - Filter (DAG–DAG) (Bielza et al., 2011)
  - Small–tw (tree–empty) (Pastink and van der Gaag, 2015)
- Performance measures:
  - Treewidth (τ)
  - Treewidth of the pruned graph (τ_p)
  - Size
  - Mean accuracy (acc_M)

Datasets

Dataset	Classes	Features	Instances
ANDES	74	82	5000
MUNIN1	62	86	5000
DIABETES	138	284	5000
MEDICAL	45	110	978
ENRON	53	124	1702

Table 2: Basic properties of the datasets

Experimental Results

Method	τ	τ_p	size	acc_M
CB–OBS	7.8±0.7	5.2±1.0	679±157	0.778±0.001*
Tree–tree	18.4±0.8	7.8±0.4	3710±724	**0.779±0.002***
Polytree–polytree	21.8±2.4	8.2±0.4	5221±342	0.777±0.003
Filter (DAG–DAG)	25.2±1.2	9.2±1.2	8756±3542	0.776±0.003
Small–tw	5.0±0.0	5.0±0.0	1382±171	0.764±0.004

Method	τ	τ_p	size	acc_M
CB–OBS	5.8±0.4	3.6±0.5	391±27	0.757±0.001
Tree–tree	14.2±1.9	7.6±1.2	2604±1226	**0.758±0.001***
Polytree–polytree	19.8±2.9	9.0±1.4	6071±4146	0.756±0.001
Filter (DAG–DAG)	21.2±1.7	8.6±1.0	5861±3918	0.756±0.001
Small–tw	5.0±0.0	5.0±0.0	1012±27	0.750±0.002

Method	τ	τ_p	size	acc_M
CB–OBS	72.2±2.9	5.4±0.5	2200±367	**0.934±0.015***
Tree–tree	66.6±10.9	37.4±2.6	$(5.183 \pm 8.999) \times 10^{12}$	–
Polytree–polytree	100.6±8.0	54.4±5.7	$(3.216 \pm 3.950) \times 10^{18}$	–
Filter (DAG–DAG)	93.0±3.2	55.8±2.5	$(1.154 \pm 1.551) \times 10^{18}$	–
Small–tw	5.0±0.0	5.0±0.0	2802±179	0.931±0.016

Table 3: Performance in ANDES (above), MUNIN1 (center), and DIABETES (below)

Experimental Results

Method	τ	τ_p	size	acc$_M$
CB–OBS	6.0±0.0	3.2±0.7	179±23	**0.988±0.002**∗
Tree–tree	13.0±1.4	5.8±1.9	802±735	0.987±0.002
Polytree–polytree	20.6±1.0	14.2±1.7	173039±182991	–
Filter (DAG–DAG)	10.0±1.3	4.0±0.9	286±52	0.987±0.002
Small–tw	3.0±0.0	2.8±0.4	250±7	0.986±0.002

Method	τ	τ_p	size	acc$_M$
CB–OBS	31.4±1.9	3.8±0.4	273±35	0.946±0.008
Tree–tree	12.0±1.1	5.4±0.5	743±85	**0.946±0.007**∗
Polytree–polytree	15.0±1.7	9.6±0.8	5025±2572	0.946±0.007
Filter (DAG–DAG)	32.8±1.8	4.6±0.5	494±61	0.945±0.010
Small–tw	5.0±0.0	5.0±0.0	671±38	0.944±0.009

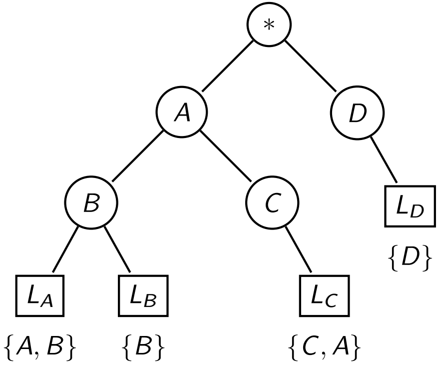
Table 4: Performance in MEDICAL (above) and ENRON (below)

Conclusions and Future Research

Conclusions

- We provided **new upper bounds** for the complexity of MBCs
- We showed that **CB-decomposability** can be used to **efficiently guarantee the tractability** of MBCs
- We proposed a learning method that learns tractable CB-decomposable MBCs
- Experimental results showed that the proposed method is **competitive** with other state-of-the-art methods in terms of **accuracy**, also ensuring that the learned models can perform multidimensional classification efficiently
- We observed that some models remain tractable even with a large treewidth

Future Research: Learning Elimination Orders



Equivalent elimination orders:

{B,C,D,A}
{B,C,A,D}
{C,B,D,A}
...

- Not restricted to **topological orders**
- Each tree represents a **set of elimination orders**
- **Reduces the redundancy** of the space of elimination orders and DAGs

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