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Incoherence Correction and Decision Making Based on Generalized Credal Sets

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The outline of the paper

- Recently the concept of generalized credal set has been proposed for modeling conflict, imprecision and contradiction in information.
- We call information contradictory if avoiding sure loss condition is violated.
- We show that any contradictory lower prevision can be represented as a convex sum of non-contradictory and fully contradictory lower previsions.
- Then we find connections with this representation with generalized credal sets.
- Based on contradiction-imprecision transformation viewed as incoherence correction we show how generalized credal sets can be applied to decision problems.

Credal Sets, Lower Previsions

- Let X be a finite set, 2^X be the powerset of X and M_{pr} be the set of all probability measures on 2^X.
- Any $P \in M_{pr}$ can be represented as a point $(P(\{x_1\}), ..., P(\{x_n\}))$ in $\mathbb{R}^n$.
- A *credal set* **P** is a non-empty subset of M_{pr}, which is convex and closed.
- Let K be a set of all real-valued functions $f : X \to \mathbb{R}$.
- Any $f \in K$ can be viewed as a random variable for a fixed $P \in M_{pr}$
- The expectation of $f \in K$ is defined by $E_P(f) = \sum_{x \in X} f(x)P(\{x\})$.

Let K' be an arbitrary subset of K, then any functional $\underline{E} : K' \to \mathbb{R}$ is called a *lower prevision* if each value $\underline{E}(f)$, $f \in K'$, is viewed as a lower bound of expectation of the random variable f. This lower prevision is called *non-contradictory* (or it avoids sure loss) iff it defines the credal set

$$\mathbf{P}(\underline{E}) = \{P \in M_{pr} | \forall f \in K' : E_P(f) \geqslant \underline{E}(f)\} \quad (1)$$

Otherwise, when the set $\mathbf{P}(\underline{E})$ is empty, the lower prevision is called *contradictory* (or *incoherent*). Analogously, upper previsions are defined.

Remark 1. Obviously,

$$\min_{x\in X} f(x) \leqslant E_P(f) \leqslant \max_{x\in X} f(x)$$

for any $P \in M_{pr}$ and $f \in K$. Thus, without decreasing generality we can assume that values $\underline{E}(f)$ of any lower prevision $\underline{E} : K' \to \mathbb{R}$ should be not larger than $\max_{x\in X} f(x)$, i.e. $\underline{E}(f) \leqslant \max_{x\in X} f(x)$ for any $f \in K'$. Analogously, we will assume that $\bar{E}(f) \geqslant \min_{x\in X} f(x)$ for any upper prevision $\bar{E} : K' \to \mathbb{R}$ and $f \in K'$. This assumption will be used later without mentioning about it.

Contradictory Lower Previsions

Definition 1. A lower prevision $\underline{E} : K' \to \mathbb{R}$ is called *fully contradictory* iff $\underline{E}$ can not be represented as a convex sum

$$\underline{E}(f) = a\underline{E}^{(1)}(f) + (1-a)\underline{E}^{(2)}(f)$$

of a non-contradictory lower prevision $\underline{E}^{(1)}$, and a (contradictory) lower prevision $\underline{E}^{(2)}$ for some $a \in (0, 1]$.

Lemma 1. *A lower prevision $\underline{E} : K' \to \mathbb{R}$ is fully contradictory iff for any $a \in (0, 1]$ the lower prevision*

$$\underline{E}'(f) = \frac{1}{a}\left(\underline{E}(f) - (1-a)\max_{x \in X} f(x)\right),$$

$f \in K'$, is contradictory.

Lemma 2. *If the set of contradictory previsions on* K' *is not empty, then the lower prevision*

$$\underline{\hat{E}}(f) = \max_{x \in X} f(x),$$

$f \in K'$, *is fully contradictory.*

Remark 2. It is possible to choose K' such that every lower prevision is non-contradictory. In this case $\underline{\hat{E}}$ is also a non-contradictory lower prevision. Because the aim of the paper is to deal with contradictory information, in the next we will assume that K' is chosen providing the lower prevision $\underline{\hat{E}}$ to be fully contradictory.

The amount of contradiction

By Lemmas 1-2 any lower prevision $\underline{E} : K' \to \mathbb{R}$ can be represented as

$$\underline{E}(f) = a\underline{E}^{(1)}(f) + (1-a)\underline{E}^{(2)}(f), \qquad (2)$$

where $\underline{E}^{(1)}$ is non-contradictory and $\underline{E}^{(2)}$ is fully contradictory. If $a \in (0,1]$, then by Lemma 1 $\underline{E}^{(2)}$ can be chosen to be equal to $\underline{\hat{E}}$.

Definition 2. The amount of contradiction in $\underline{E}$ is defined by

$$Con(\underline{E}) = 1 - \sup\{a | a \in A\},$$

where A is the set of all possible a satisfying (2).

Generalized Credal Sets

Consider monotone measures, viewed as lower probabilities, on 2^X of the type

$$P = a_0 \eta^d_{\langle X \rangle} + \sum_{i=1}^{n} a_i \eta_{\langle \{x_i\} \rangle},$$

where $\sum_{i=0}^{n} a_i = 1$, $a_i \geqslant 0$, $i = 0, \ldots, n$, and

$$\eta^d_{\langle X \rangle}(A) = \begin{cases} 1, & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases} \quad \eta_{\langle \{x_i\} \rangle}(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Clearly, $P = a_0 \eta^d_{\langle X \rangle} + (1 - a_0) P'$,

where $\eta^d_{\langle X \rangle}$ is a fully contradictory lower probability

and $P' = \frac{1}{1-a_0} \sum_{i=1}^{n} a_i \eta_{\langle \{x_i\} \rangle}$ is a probability measure.

We can extend P to the lower prevision on the set of all functions in K by

$$\underline{E}_P(f) = a_0 \max_{x \in X} f(x) + \sum_{i=1}^{n} a_i f(x_i).$$

Again $\underline{E}_P$ can be represented as a convex sum of fully contradictory lower prevision $\hat{\underline{E}}$ and linear prevision $E_{P'}$, i.e.

$$\underline{E}_P(f) = a_0 \hat{\underline{E}}(f) + (1 - a_0) E_{P'}(f) \text{ for all } f \in K.$$

The set of all

$$P = a_0 \eta^d_{\langle X \rangle} + (1 - a_0) P',$$

where $P' \in M_{pr}$, is denoted by M_{cpr}.

Lemma 3. *Let* $P = a_0\eta^d_{\langle X\rangle} + \sum_{i=1}^{n} a_i\eta_{\langle\{x_i\}\rangle}$ *be in* M_{cpr}*.* *Then* $Con(P) = a_0$.

We will identify each $P \in M_{cpr}$ with a point $(a_1, ..., a_n)$ in $\mathbb{R}^n$.

Definition 3. A subset $\mathbf{P}$ of M_{cpr} is called an *upper generalized credal set* (UG-credal set) if

1. $P_1 \in \mathbf{P}$, $P_2 \in M_{cpr}$, and $P_1(A) \leqslant P_2(A)$ for all $A \in 2^X$ implies $P_2 \in \mathbf{P}$;
2. $\mathbf{P}$ is a convex closed set as a subset of $\mathbb{R}^n$.

We will describe any lower prevision $\underline{E} : K' \to \mathbb{R}$ by a UG-credal set P defined by

$$\mathbf{P} = \{P \in M_{cpr} | \forall f \in K' : \underline{E}(f) \leqslant \underline{E}_P(f)\} . \quad (3)$$

Remark 3. Obviously, the set defined by (3) is not empty, because it always contains the measure $\eta^d_{\langle X \rangle}$.

Proposition 1. *Let $\underline{E} : K' \to \mathbb{R}$ be a lower prevision, and let P be its corresponding UG-credal set defined by (3). Then*

$$Con(\underline{E}) = \inf \{Con(P) | P \in \mathbf{P}\} . \quad (4)$$

Some details

Let $X = \{x_1, x_2, ..., x_n\}$, then any $P \in M_{cpr}$ is represented by

$$P = a_0 \eta_{\langle X \rangle} + a_1 \eta_{\langle \{x_1\} \rangle} + ... + a_n \eta_{\langle \{x_n\} \rangle}$$

$$P(A) = \begin{cases} a_0 + \sum_{x_i \in A} a_i, & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases}$$

Let $P_1 = (a_1^{(1)}, ..., a_n^{(1)})$ and $P_2 = (a_1^{(2)}, ..., a_n^{(2)})$. Then $P_1(A) \geqslant P_2(A)$ for all $A \in 2^X$ iff $a_i^{(1)} \leqslant a_i^{(2)}$, $i = 1, ..., n$.

Some details

Let $X = \{x_1, x_2\}$, then any $P \in M_{cpr}$ is a point within the triangle $\{(a_1, a_2) | a_1 \geqslant 0, a_2 \geqslant 0, a_1 + a_2 \leqslant 1\}$ (see Fig.1). The minimal UG-credal set $\mathbf{P}$, containing P, is $\mathbf{P} = \{P' \in M_{cpr} | P' \geqslant P\}$ (yellow rectangle on Fig.1).

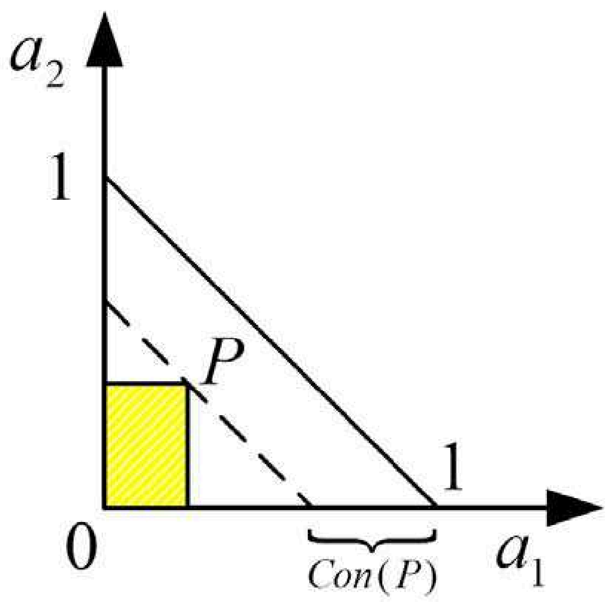


Fig. 1.

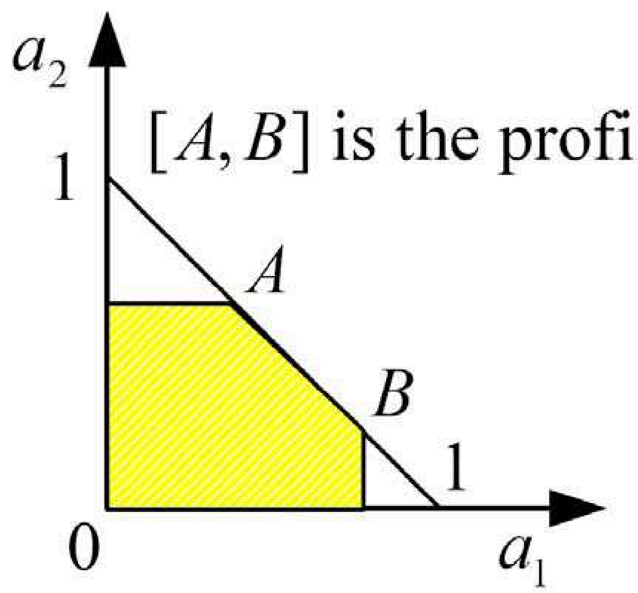
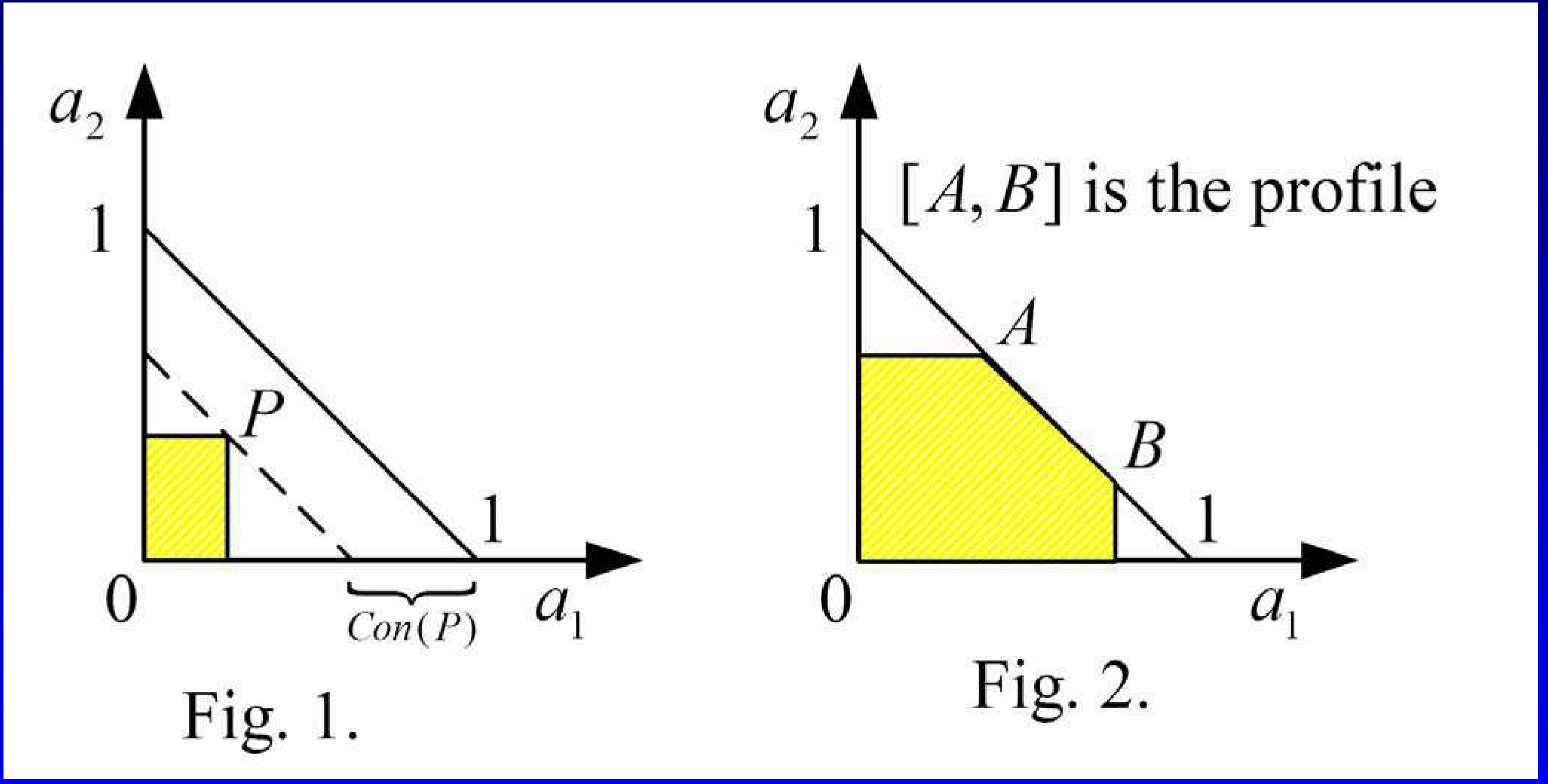


Fig. 2.

Some details

The set of all minimal elements in an UG-credal set is called the *profile*. If information is described by usual credal set $\mathcal{P} \subseteq M_{pr}$, then we can describe this information by UG-credal set, whose profile is $\mathcal{P}$, as shown on Fig. 2.



Incoherence Correction

- Assume that $\underline{E} : K' \to \mathbb{R}$ is a lower prevision and $Con(\underline{E}) = b$.
- If $b = 1$ then $\underline{E}$ is fully contradictory and $\underline{E}$ does not contain useful information and this case can be characterized as full ignorance.
- Let $b < 1$, then our lower prevision can be represented as

$$\underline{E}(f) = (1 - b)\underline{E}^{(1)}(f) + b\hat{\underline{E}}(f),$$

$f \in K'$, and we should use information in $\underline{E}^{(1)}$ for choosing decisions.

Assume that a non-contradictory lower prevision $\underline{E}^{(1)}$ defines the credal set

$$\mathbf{P}' = \left\{ P \in M_{pr} | \forall f \in K' : \underline{E}^{(1)}(f) \leqslant E_P(f) \right\}.$$

Then taking in account that $\hat{\underline{E}}$ describes the case of full contradiction, we can describe $\underline{E}$ by a credal set $\mathbf{P}''$ represented as a convex sum of two credal sets $\mathbf{P}'$ and M_{pr}, where M_{pr} describes the case of full ignorance:

$$\mathbf{P}'' = \{(1-b)P_1 + bP_2 | P_1 \in \mathbf{P}', P_2 \in M_{pr}\}. \quad (5)$$

The following proposition shows how the above set $\mathbf{P}''$ can be found based on UG-credal sets.

Proposition 2. *Let* $\underline{E} : K' \to \mathbb{R}$ *be a lower prevision,* $Con(\underline{E}) = b$*, and let* $\mathbf{P}$ *be its corresponding UG-credal set. Then*

$$\mathbf{P}'' = \{P' \in M_{pr} | \exists P \in \mathbf{P} : Con(P) = b, P' \leqslant P\}.$$

The above transformation of a contradictory lower prevision to the non-contradictory information can be considered as incoherence correction in which full contradiction is transformed to full ignorance.

Decision Making

After contradiction-imprecsion transformation we can use known models of decision making considered in imprecise probabilities.

Example 1. Given two pieces of evidence:

- "probability of sunny $\geqslant 0.3$";
- "probability of rain $\geqslant 0.8$".

Denote $x_1 := sunny$, $x_2 := rain$, $X = \{x_1, x_2\}$.

This information is described by UG-credal set

$$\mathbf{P} = \{P \in M_{cpr} | P(\{x_1\}) \geqslant 0.3, P(\{x_1\}) \geqslant 0.8\},$$

where $P = a_0 \eta^d_{\langle X \rangle} + a_1 \eta_{\langle\{x_1\}\rangle} + a_2 \eta_{\langle\{x_1\}\rangle}$.

Thus, the set $\mathbf{P}$ is described by the following inequalities:

$$\left\{\begin{array}{c} a_1 + a_0 \geqslant 0.3 \\ a_2 + a_0 \geqslant 0.8 \\ a_1 + a_2 + a_0 = 1 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} a_1 \leqslant 0.2 \\ a_2 \leqslant 0.7 \\ a_1 + a_2 + a_0 = 1 \end{array}\right.$$

The only element in $\mathbf{P}$ with the minimal contradiction is

$$P = 0.1\eta^d_{\langle X\rangle} + 0.2\eta_{\langle\{x_1\}\rangle} + 0.7\eta_{\langle\{x_2\}\rangle}.$$

The contradiction-imprecision transformation gives us the credal set

$$\mathbf{P}'' = \{P' \in M_{pr} | P' \leqslant P\}.$$

Then, clearly,

$$\underline{E}_{\mathbf{P}''}(f) = 0.2f(x_1) + 0.7f(x_2) + 0.1 \min_{x \in X} f(x).$$

Assume, for example, that we have two decisions:

- $g_1 := \textit{go to the park}$ ($g_1(x_1) = 3$, $g_1(x_2) = -1$);
- $g_2 := \textit{go to the theater}$ ($g_2(x_1) = 1$, $g_2(x_2) = 1$).

Then

$$\underline{E}_{\mathbf{P}''}(g_2 - g_1) = 0.2 \cdot (-2) + 0.7 \cdot 2 + 0.1 \cdot (-2) = 0.8 > 0,$$

i.e. decision g_2 is more preferable than decision g_1.

Thank you for attention!!!