On Boolean algebras of conditionals and their logical counterpart

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Motivation

- Conditionals play a fundamental role both in qualitative (non-monotonic) and quantitative uncertain reasoning (conditional measures)
- In the literature, conditionals are usually regarded as 3-valued objects
- In a previous Ecsqaru-15 paper, we argued for an approach to deal with Boolean conditionals, motivated by the relationship between conditional probabilities and probabilities on conditional objects
- In this paper we make two main contributions:
  1. the atomic structure of Boolean algebras of conditionals
  2. a corresponding logic to reason with conditionals

Boolean Algebras of Conditionals

The starting point of our investigation was to propose an algebraic construction that, given a boolean algebra $\mathbf{A}$ of events produces another algebra of "conditional events" $\mathcal{C}(\mathbf{A})$ fulfilling the following conditions:

- $\mathcal{C}(\mathbf{A})$ is a Boolean algebra
- $\mathcal{C}(\mathbf{A})$ is finite whenever $\mathbf{A}$ is finite
- Allowing to relate simple measures on $\mathcal{C}(\mathbf{A})$ and conditional measures on $\mathbf{A}$

The idea is not new, see for instance

- Goodman-Nguyen-van Fraassen approach (infinite boolean algebra of conditional events).
- Calabrese: the algebra of events is not distributive.
- In general the literature on Conditional Events Algebras (CEA).
- ...
- Gilio and Sanfilippo, Schay, Gehrke and Walker...

Boolean Algebras of Conditionals

$\mathbf{A} = (A, \wedge, \vee, \neg, \bot)$: a *finite* Boolean algebra of events,
$A' = A \setminus \{\bot\}$

Construction of the corresponding algebra of conditionals (intuitively):

1. take all pairs $(a, b) \in A \times A'$
2. consider all free combinations with Boolean operations $\wedge^*, \vee^*, \neg^*$:
   $((a, b) \wedge^* (c, d)) \vee^* (e, f), \neg^*(a, b), \ldots \in \mathcal{F}(A \times A')$
3. identify desirable equivalences for conditionals:
   $(a, b) \equiv (a \wedge b, b)$, $(a, b) \wedge^* (c, b) \equiv (a \wedge c, b), \ldots$
4. define conditionals as equivalence classes $\mathcal{F}(A \times A')$:
   $(a \mid b) = \{(x, y) : (x, y) \equiv (a, b)\}$
5. endow $\mathcal{F}(A \times A')/\equiv$ with Boolean operations $\sqcap, \sqcup, \neg$ compatible with $\wedge^*, \vee^*, \neg^*$:
   $(a \mid b) \sqcap (c \mid d) = \{(a', b') \wedge^* (c', d') : (a', b') \wedge^* (c', d') \equiv (a, b) \wedge^* (c, d)\}$
   $\ldots$

Boolean Algebras of Conditionals

$\mathbf{A} = (A, \wedge, \vee, \neg, \bot)$: a *finite* Boolean algebra of events,
$A' = A \setminus \{\bot\}$

Construction of the corresponding algebra of conditionals (algebraically):

1. take the free Boolean algebra $\mathcal{F}(A \times A')$
2. let $a_1 \Leftrightarrow a_2 := (\neg^* a_1 \vee^* a_2) \wedge^* (\neg^* a_2 \vee^* a_1)$, for $a_1, a_2 \in \mathcal{F}(A \times A')$.
3. consider the filter $\mathfrak{F}$ of $\mathcal{F}(A \times A')$ generated by the following elements, for every $x, z \in A$ and $y, w \in A'$:

   (t1) (y, y)
   (t2) $(x, y) \wedge^* (z, y) \Leftrightarrow (x \wedge z, y)$
   (t3) $\neg^*(x, y) \Leftrightarrow (\neg x, y))$
   (t4) $(x \wedge y, y) \Leftrightarrow (x, y)$
   (t5) $(x, w) \Leftrightarrow (x, y) \wedge^* (y, w)$, if $x \leq y \leq w$

4. define the congruence relation $\equiv_{\mathfrak{F}}$ in $\mathcal{F}(A \times A')$ as

$$c \equiv_{\mathfrak{F}} c' \quad \text{if} \quad c \Leftrightarrow c' \in \mathfrak{F}.$$

Boolean Algebras of Conditionals

DEFINITION. The *Boolean algebra of conditionals* of **A** (the *BAC-algebra* of **A**, for short) is the quotient algebra

$$\mathcal{C}(\mathbf{A}) = \mathcal{F}(A \times A')/_{\equiv_{\mathfrak{F}}}$$

Notice that:

- $\mathcal{C}(\mathbf{A})$ is a finite Boolean algebra, and hence $\mathcal{C}(\mathbf{A})$ is atomic.
- The elements of $\mathcal{C}(\mathbf{A})$ are equivalence classes (modulo $\equiv_{\mathfrak{F}}$) of Boolean terms on variables in $A \times A'$. We will write $(a \mid b)$ instead of $[(a, b)]_{\equiv_{\mathfrak{F}}}$ for the *basic conditionals*.
- **A** is isomorphic to the subalgebra of $\mathcal{C}(\mathbf{A})$ of elements $(a \mid \top)$ for $a \in A$
- Not every element of $\mathcal{C}(\mathbf{A})$ can be written in the form $(a \mid b)$. For instance,
  - $\mathcal{C}(\mathbf{A}) \models (a \mid b) \sqcap (c \mid b) = (a \wedge c \mid b)$, but
  - in general, if $b_1 \neq b_2$, for all $x \in A, y \in A'$ s.t. $\mathcal{C}(\mathbf{A}) \not\models (a \mid b_1) \sqcap (c \mid b_2) = (x \mid y)$

The atoms of $\mathcal{C}(\mathbf{A})$

THE ATOMS OF $\mathcal{C}(\mathbf{A})$

An element α of **A** is an *atom* if α is non-$\perp$ and minimal w.r.t. to the lattice order of **A**, i.e., $\perp < \alpha$ and, if $\perp \leq x \leq \alpha$, then either $x = \perp$, or $x = \alpha$.

Every finite Boolean algebra **A** is atomic and atoms play several roles in logic and algebra:

- Atoms corresponds bijectively to $\{0, 1\}$-valued logical valuations;
- Atoms corresponds bijectively to maximally consistent theories;
- Every element of **A** is uniquely representable as a disjunction of atoms.

Therefore, knowing $Atom(\mathbf{A})$, gives many information about **A** itself.

Is it possible to characterize $Atom(\mathcal{C}(\mathbf{A}))$ knowing $Atom(\mathbf{A})$?

THE ATOMS OF $\mathcal{C}(\mathbf{A})$

Consider an algebra **A** with n atoms, say

$$Atom(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}.$$

We proved that the atoms of $\mathcal{C}(\mathbf{A})$ are elements of the form:

- Take $Seq(\mathbf{A}) = \{\overline{\alpha} = (\alpha_{\iota(1)}, \ldots, \alpha_{\iota(n-1)}) \mid \alpha_{\iota(j)} \in Atom(\mathbf{A})\}$,
- Define

$$\omega_{\overline{\alpha}} = (\alpha_{\iota(1)} \mid \top) \sqcap (\alpha_{\iota(2)} \mid \neg\alpha_{\iota(1)}) \sqcap \ldots \sqcap (\alpha_{\iota(n-1)} \mid \neg\alpha_{\iota(1)} \wedge \ldots \wedge \neg\alpha_{\iota(n-2)}).$$

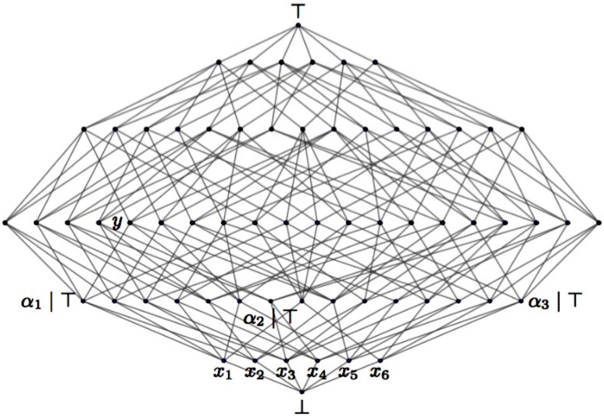
THEOREM. Let **A** be a Boolean algebra with n atoms and let Seq(**A**) the set of sequences of $n-1$ atoms of **A**. Then, the set of the atoms of $\mathcal{C}(\mathbf{A})$ is

$$Atom(\mathcal{C}(\mathbf{A})) = \{\omega_{\overline{\alpha}} : \overline{\alpha} \in Seq(\mathbf{A})\}.$$

As a consequence, $|Atom(\mathcal{C}(A))| = n!$ and $|\mathcal{C}(A)| = 2^{n!}$

EXAMPLE OF $\mathcal{C}(\mathbf{A})$

$Atom(\mathbf{A}) = \{\alpha_1, \alpha_2, \alpha_2\}$
$Atom(\mathcal{C}(\mathbf{A})) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$,
The element $y = (\alpha_1 \mid \neg\alpha_3)$ is $x_1 \sqcup x_2 \sqcup x_5$



The logic of Boolean conditionals LBC

LBC: THE LOGIC OF BOOLEAN CONDITIONALS

In classical logic, truth-evaluations of variables are in 1-1 relation with the atoms of the Lindenbaum's algebra of propositions (finite case).

The previous characterization of $Atom(\mathcal{C}(\mathbf{A}))$ thus suggests a natural way to define truth-evaluation of conditional objects as logical formulas.

The language $\mathcal{CL}$:

- Fix a finite set V of propositional variables and build formulas $\varphi, \psi, \gamma, \ldots$ as usual in classical propositional logic.
- For every pair of classical formulas φ, ψ such that $\not\vdash \psi \to \bot$, $(\varphi \mid \psi)$ is a *basic conditional formula*.
- The set of $\mathcal{CL}$-formulas is finally obtained as the closure, under Boolean connectives, of basic conditional formulas.

Let $|V| = m$. Call Ω the set of its $k = 2^m$ possible truth-evaluations $w_1, \ldots, w_k$.

Semantics of LBC

DEFINITION

- A $\mathcal{CL}$*-evaluation* is an ordered sequence $e = (w_1, \ldots, w_k)$ of k pairwise distinct evaluations in Ω.
- For a formula $\Phi \in \mathcal{CL}$ and a $\mathcal{CL}$-evaluation $e = (w_1, \ldots, w_k)$, we say that Φ *is true in* e (and we write $e \models \Phi$) by induction as follows:

  (1) If $\Phi = (a \mid b)$ is atomic, $e \models (a \mid b)$ if

  $$\text{for some } 1 \leq i \leq k-1, w_i(a \wedge b) = 1 \text{ and } w_j(b) = 0 \text{ for all } j < i.$$

  This means, $e \models (a \mid b)$ if either
  $w_1(a \wedge b) = 1$, or
  $w_1(b) = 0$ and $w_2(a \wedge b) = 1$, or
  $w_1(b) = w_2(b) = 0$ and $w_3(a \wedge b) = 1$, or
  $\ldots$

  (2) If Φ is compound, then $e \models \Phi$ is determined by (1) and the truth functionality of Boolean connectives.

- A $\mathcal{CL}$-formula Φ is said to be $\mathcal{CL}$*-valid* (and we write $\models_{\mathcal{CL}} \Phi$), if $e \models \Phi$ for every $\mathcal{CL}$-evaluation e.

Axiomatization of LBC

The *Logic of Boolean conditionals* (LBC for short) has the following axioms and rules, where $\vdash_{PL}$ denotes classical derivability:

(PL) Axioms and rule of classical propositional logic for $\mathcal{CL}$ formulas

(A1) $(\psi \mid \psi)$

(A2) $\neg(\varphi \mid \psi) \leftrightarrow (\neg\varphi \mid \psi)$

(A3) $(\varphi \mid \psi) \wedge (\delta \mid \psi) \leftrightarrow (\varphi \wedge \delta \mid \psi)$

(A4) $(\varphi \mid \psi) \leftrightarrow (\varphi \wedge \psi \mid \psi)$

(A5) $(\varphi \mid \psi) \leftrightarrow (\varphi \mid \chi) \wedge (\chi \mid \psi)$, if $\vdash_{PL} \varphi \to \chi$ and $\vdash_{PL} \chi \to \psi$

(R1) from $\vdash_{PL} \varphi \to \psi$ derive $(\varphi \mid \chi) \to (\psi \mid \chi)$

(R2) from $\vdash_{PL} \chi \leftrightarrow \psi$ derive $(\varphi \mid \chi) \leftrightarrow (\varphi \mid \psi)$

The notion of proof in LBC, $\vdash_{LBC}$, is defined as usual.

THEOREM. LBC is sound and complete w.r.t. $\mathcal{CL}$-evaluations, i.e.

$$\vdash_{LBC} = \models_{LBC} .$$

NONMONOTONIC REASONING

Conditionals have an implicit non-monotonic behaviour. Given a conditional $(\varphi \mid \psi)$, it does not follow in general that we can freely strengthen its antecedent, i.e. in general, $(\varphi \mid \psi) \nvdash_{LBC} (\varphi \mid \psi \wedge \chi)$.

Now, let us fix a set of (atomic) conditional statements K, and let us define the consequence relation associated to K:

$$\varphi \mid\!\sim_K \psi \ \text{ if } \ K \vdash_{LBC} (\psi \mid \varphi)$$

PROPOSITION $\mid\!\sim_K$ is a preferential consequence relation.

Indeed, $\vdash_{LBC}$ satisfies the following related properties:

Reflexivity: $\vdash_{LBC} (\varphi \mid \varphi)$
Left logical equivalence: if $\models_{PL} \varphi \leftrightarrow \psi$ then $(\chi \mid \varphi) \vdash_{LBC} (\chi \mid \psi)$
Right weakening: if $\models_{PL} \varphi \rightarrow \psi$ then $(\varphi \mid \chi) \vdash_{LBC} (\psi \mid \chi)$
Cut: $(\varphi \mid \psi) \wedge (\chi \mid \varphi \wedge \psi) \vdash_{LBC} (\chi \mid \psi)$
Or: $(\varphi \mid \psi) \wedge (\varphi \mid \chi) \vdash_{LBC} (\varphi \mid \psi \vee \chi)$
And: $(\varphi \mid \psi) \wedge (\delta \mid \psi) \vdash_{LBC} (\varphi \wedge \delta \mid \psi)$
Cautious Monotony: $(\varphi \mid \psi) \wedge (\chi \mid \psi) \vdash_{LBC} (\chi \mid \varphi \wedge \psi)$

Future work: (i) LBC and 3-valued semantics

Given a $\mathcal{CL}$*-evaluation* $e = (w_1, \ldots, w_k)$ with $k = 2^m$ pairwise distinct propositional evaluations from Ω, we can consider the succession $[w_1], [w_1, w_2], [w_1, w_2, w_3] \ldots$ of partial sequences of increasing length.

Each partial sequence $[w_1, \ldots, w_n]$, with $n < k$, can be seen as an approximation of the original $\mathcal{CL}$-evaluation e, and defines a 3-valued evaluation of atomic conditionals:

$$[w_1, \ldots, w_n](a \mid b) = \begin{cases} 1, & \text{if } \exists i \leq n(w_i(a \wedge b) = 1 \text{ and } w_j(b) = 0 \text{ for all } j < i), \\ 0, & \text{if } \exists i \leq n(w_i(\neg a \wedge b) = 1 \text{ and } w_j(b) = 0 \text{ for all } j < i), \\ u, & \text{otherwise} \end{cases}$$

Now, all partial sequences of a given length, say n, determine a 3-valued semantics for conditionals.

The usual 3-valued semantics for conditionals in the literature can be recovered when taking $n = 1$.

FUTURE WORK: (II) PROBABILITIES IN $\mathcal{C}(\mathbf{A})$

Our original motivation: relationship between conditional probabilities on $\mathbf{A}$ and plain probabilities on $\mathcal{C}(\mathbf{A})$

Question: for any probability $P : \mathbf{A} \to [0, 1]$, is there a probability $\mu : \mathcal{C}(\mathbf{A}) \to [0, 1]$ such that $P(a \mid b) = \mu(``(a \mid b)'')$?

- No formal proof, but the conjecture is Yes
- Given $P : \mathbf{A} \to [0, 1]$, and $\omega_{\overline{\alpha}} = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2})$, define

$$\mu(\omega_{\overline{\alpha}}) = P(\alpha_1) \cdot P(\alpha_2 \mid \alpha_1) \cdot \ldots \cdot P(\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2})$$

- Analytically proved for $n = 3, 4$ and computational evidence for $n = 5, 6, 7$

Thank you