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An Angel-Daemon Approach to Assess the Uncertainty in the Power to Act

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Introduction

Risk versus Uncertainty

Frank Knight. Risk Uncertainty and Profit, 1921.

Distinction between risk and uncertainty:

- Risk refers to something that can be measured by mathematical probabilities
- Uncertainty refers to something that cannot be measured because there are no objective standards to express probabilities

Uncertainty

We analyze uncertainty through strategic situations.

An uncertainty profile $\mathcal{U}$ gives,

- a short and macroscopic description of the potential stress of a system,
- together with the a description of an strategic situation.

In this strategic situation, two agents, the angel $\mathfrak{a}$ and the daemon $\mathfrak{d}$ have opposite goals.

This work: Assess the Uncertainty in the

Power of a Collectivity to Act

Simple games and Weighted voting games

Simple games

In simple game $\Gamma = (N, \mathcal{W})$,

- N is a set of n players,
- $\mathcal{W}$ is a monotonic family of subsets of N.

The subsets of N are called coalitions:

- N is the grand coalition and
- $S \in \mathcal{W}$ is a winning coalition,
- Any not winning subset of N is a losing coalition.

The Coleman's power of the collectivity to act is:

$$\text{Act}(\Gamma) = \#\mathcal{W}/2^n$$

It can be seen as the probability of the yes outcome assuming that all coalitions are equally like.

Weighted voting games

A weighted voting game is a simple game defined by a tuple $\Gamma = \langle q; w_1, \ldots, w_n \rangle$, where

- $N = [n] = \{1, \ldots, n\}$ is the set of players,
- q is the quota and
- $w_i \in \mathbb{N}^+$ is the weight of player i, for all $1 \leq i \leq n$.

Let $w(S) = \sum_{i \in S} w_i$ denote the weight of coalition S.

- The set of winning coalitions is

$$\mathcal{W}(\Gamma) = \{S \mid w(S) \geq q\}$$

- the set of losing coalitions is

$$\mathcal{L}(\Gamma) = \{S \mid w(S) < q\}$$

Example 1: The Council of the EU at 1958

In non-decreasing order of assigned weights.

$$\{\text{DE}, \text{FR}, \text{IT}, \text{NL}, \text{BE}, \text{LU}\} = \{1, 2, 3, 4, 5, 6\}$$

The Council is summarized as

$$\begin{aligned}\Gamma_{\text{EC6}} &= \langle q; w_1, w_2, w_3, w_4, w_5, w_6 \rangle \\ &= \langle 12; 4, 4, 4, 2, 2, 1 \rangle\end{aligned}$$

$\#\mathcal{W}([6]) = 17$, the quota was a majority of the 70.6%

$$q = 12 \approx 17 * (70.6/100) = 12.002$$

Succinct notation $\Gamma_{\text{EC6}} = \langle 12; 3{:}4, 2{:}2, 1{:}1 \rangle$

$$\text{Act}(\Gamma_{\text{EC6}}) = \#\mathcal{W}(\Gamma_{\text{EC6}})/2^n = 14/2^6 = 0.2187$$

Council of the EU along the time (1958-2014)

$$\Gamma_{EC6} = \langle 12; 3{:}4, 2{:}2, 1{:}1 \rangle$$
$$\Gamma_{EC9} = \langle 41; 3{:}10, 2{:}5, 1{:}2 \mid 1{:}10, 2{:}3 \rangle$$
$$\Gamma_{EC10} = \langle 45; 3{:}10, 2{:}5, 1{:}2 \mid 1{:}10, 1{:}5, 2{:}3 \rangle$$
$$\Gamma_{EC12} = \langle 54; 3{:}10, 2{:}5, 1{:}2 \mid 1{:}10, 1{:}8, 2{:}5, 2{:}3 \rangle$$
$$\Gamma_{EC15} = \langle 62; 3{:}10, 2{:}5, 1{:}2 \mid 1{:}10, 1{:}8, 2{:}5, 2{:}4, 3{:}3 \rangle$$
$$\Gamma_{EC25_1} = \langle 88; 3{:}10, 2{:}5, 1{:}2 \mid 1{:}10, 2{:}8, 4{:}5, 2{:}4, 8{:}3, 2{:}2 \rangle$$
$$\Gamma_{EC25_2} = \langle 232; 3{:}29, 1{:}13, 1{:}12, 1{:}4 \mid 1{:}29, 2{:}27, 4{:}12, 2{:}10, 5{:}7, 4{:}4, 1{:}3 \rangle$$
$$\Gamma_{EC27} = \langle 255; 3{:}29, 1{:}13, 1{:}12, 1{:}4 \mid 1{:}29, 2{:}27, 1{:}14, 4{:}12, 3{:}10, 5{:}7, 4{:}4, 1{:}3 \rangle$$

Γ	Γ_{EC6}	Γ_{EC9}	Γ_{EC10}	Γ_{EC12}	Γ_{EC15}	Γ_{EC25_1}	Γ_{EU25_2}	Γ_{EC27}
$w([n])$	17	58	63	76	87	124	321	345
% of q	70.6	70.7	71.4	71.1	71.3	71	72.3	73.9
$\#\mathcal{W}(\Gamma)$	14	75	140	402	2549	1170000	1204448	2718774
Act(Γ)	0.2187	0.1464	0.1455	0.0981	0.0777	0.0348	0.0358	0.0202

Uncertainty profiles and $\mathfrak{a}/\mathfrak{d}$ games

Uncertainty Profile

$\mathcal{U} = \langle \Gamma, \mathcal{A}, \mathcal{D}, \delta_{\mathfrak{a}}, \delta_{\mathfrak{d}}, b_{\mathfrak{a}}, b_{\mathfrak{d}} \rangle$:

- $\Gamma = \langle q; w_1, \ldots, w_n \rangle$ is a weighted voting game;
- $\mathcal{A}, \mathcal{D} \subseteq [n]$ are the sets of players whose weights may be subject to angelic and daemonic perturbations;
- $\delta_{\mathfrak{a}} : \mathcal{A} \to \mathbb{Z}$ and $\delta_{\mathfrak{d}} : \mathcal{D} \to \mathbb{Z}$ represent the strength of the potential weight's perturbations;
- $b_{\mathfrak{a}}$, $b_{\mathfrak{d}} \in \mathbb{N}$ such that $b_{\mathfrak{a}} \leq \#\mathcal{A}$ and $b_{\mathfrak{d}} \leq \#\mathcal{D}$. They represent the spread of the perturbations.

Remind: a short and macroscopic description of the potential stress of a system

Perturbed game $\Gamma[a, d]$

Given (a, d), for $a \subseteq \mathcal{A}$, $d \subseteq \mathcal{D}$, the perturbed game

$$\Gamma[a, d] = \langle q; w'_1, \ldots, w'_n \rangle$$

is defined as

$$w'_i = \begin{cases} w_i \;\text{ if } i \notin a \cup d \\ w_i + \delta_{\mathfrak{a}}(i) \;\text{ if } i \in a \setminus d \\ w_i + \delta_{\mathfrak{d}}(i) \;\text{ if } i \in d \setminus a \\ w_i + \delta_{\mathfrak{a}}(i) + \delta_{\mathfrak{d}}(i) \;\text{ if } i \in a \cap d \end{cases}$$

Remark: To ensure $w'_i \in \mathbb{N}^+$ we ask

$$|\delta_{\mathfrak{a}}(i)|, |\delta_{\mathfrak{d}}(i)|, |\delta_{\mathfrak{a}}(i) + \delta_{\mathfrak{d}}(i)| < w_i$$

Example 2

$\text{EC6} = \{\text{DE}, \text{FR}, \text{IT}, \text{NL}, \text{BE}, \text{LU}\} = \{1, 2, 3, 4, 5, 6\}$.
$\Gamma_{\text{EC6}} = \langle 12; w_1, w_2, w_3, w_4, w_5, w_6 \rangle = \langle 12; 4, 4, 4, 2, 2, 1 \rangle$.

Let $\mathcal{U} = \langle \Gamma_{\text{EC6}}, \mathcal{A}, \mathcal{D}, \delta_{\mathfrak{a}}, \delta_{\mathfrak{d}}, b_{\mathfrak{a}}, b_{\mathfrak{d}} \rangle$ where:

- $\mathcal{A} = \mathcal{D} = \{2, 3\} = \{\text{FR}, \text{IT}\}$.
- $\delta_{\mathfrak{a}}(2) = \delta_{\mathfrak{a}}(3) = 1$ and $\delta_{\mathfrak{d}}(2) = \delta_{\mathfrak{d}}(3) = -2$.
- $b_{\mathfrak{a}} = 1$, $b_{\mathfrak{d}} = 1$

Consider $(a, d) = (\{\text{IT}\}, \{\text{FR}\})$, the perturbed games is:

$$
\begin{aligned}
\Gamma_{\text{EC6}}[a, d] &= \Gamma_{\text{EC6}}[\{\text{IT}\}, \{\text{FR}\}] \\
&= \langle 12; w'_1, w'_2, w'_3, w'_4, w'_5, w'_6 \rangle \\
&= \langle 12; w, w_2 + \delta_{\mathfrak{d}}(2), w_3 + \delta_{\mathfrak{a}}(3), w_4, w_5, w_6 \rangle \\
&= \langle 12; 4, 4 - 2, 4 + 1, 2, 2, 1 \rangle \\
&= \langle 12; 4, 2, 5, 2, 2, 1 \rangle
\end{aligned}
$$

$\mathfrak{a}/\mathfrak{d}$-game $G(\mathcal{U})$

Given $\mathcal{U} = \langle \Gamma, \mathcal{A}, \mathcal{D}, \delta_{\mathfrak{a}}, \delta_{\mathfrak{d}}, b_{\mathfrak{a}}, b_{\mathfrak{d}} \rangle$, the associated angel/daemon (or $\mathfrak{a}/\mathfrak{d}$) game is $G(\mathcal{U}) = \langle \{\mathfrak{a}, \mathfrak{d}\}, A_{\mathfrak{a}}, A_{\mathfrak{d}}, u_{\mathfrak{a}}, u_{\mathfrak{d}} \rangle$ such that:

- $G(\mathcal{U})$ has two players: the angel $\mathfrak{a}$ and the daemon $\mathfrak{d}$.
- The player's actions are

$$A_{\mathfrak{a}} = \{a \subseteq \mathcal{A} \mid \#a = b_{\mathfrak{a}}\}, A_{\mathfrak{d}} = \{d \subseteq \mathcal{D} \mid \#d = b_{\mathfrak{d}}\}$$

- For $(a, d) \in A_{\mathfrak{a}} \times A_{\mathfrak{d}}$ utilities are

$$u_{\mathfrak{a}}(a, d) = \#\mathcal{W}(\Gamma[a, d]) = -u_{\mathfrak{d}}(a, d)$$

Remind: together with the a description of an strategic situation.

Act($\mathcal{U}$)

Remind. All Nash equilibria of the zero-sum game $G(\mathcal{U})$ have the same value (J. von Neumann, O. Morgenstern, 1953),

$$\nu(G(\mathcal{U})) = \max_{\alpha\in\Delta_{\mathfrak{a}}} \min_{\beta\in\Delta_{\mathfrak{d}}} \#\mathcal{W}(\alpha,\beta) = \min_{\beta\in\Delta_{\mathfrak{d}}} \max_{\alpha\in\Delta_{\mathfrak{a}}} \#\mathcal{W}(\alpha,\beta)$$

Given Γ with n players, $\mathcal{U} = \langle \Gamma, \mathcal{A}, \mathcal{D}, \delta_{\mathfrak{a}}, \delta_{\mathfrak{d}}, b_{\mathfrak{a}}, b_{\mathfrak{d}} \rangle$ and

$$u_{\mathfrak{a}}(a,d) = \#\mathcal{W}(\Gamma[a,d])$$

We define $\#\mathcal{W}(\mathcal{U}) = \nu(G(\mathcal{U}))$ and

$$\mathrm{Act}(\mathcal{U}) = \#\mathcal{W}(\mathcal{U})/2^n$$

Example 3

We continue with $\mathcal{U}$ given in Example 2.
As $\mathcal{A} = \mathcal{D} = \{\text{FR}, \text{IT}\} = \{2, 3\}$ and $b_{\mathfrak{a}} = b_{\mathfrak{d}} = 1$ we have

$$A_{\mathfrak{a}} = A_{\mathfrak{d}} = \{\{\text{FR}\}, \{\text{IT}\}\}$$

For example,

$$\begin{aligned} u_{\mathfrak{a}}(\{\text{FR}\}, \{\text{FR}\}) &= \#\mathcal{W}(\Gamma_{\text{EC6}}[\{\text{FR}\}, \{\text{FR}\}] \\ &= \#\mathcal{W}(\langle 12; 4, 3, 4, 2, 2, 1\rangle) \end{aligned}$$

The $\mathfrak{a}/\mathfrak{d}$-game is described by the following utility matrix for $\mathfrak{a}$.

	{FR}	{IT}
{FR}	$\#\mathcal{W}(\langle 12; 2{:}4, 1{:}3, 2{:}2, 1{:}1\rangle) = 11$	$\#\mathcal{W}(\langle 12; 1{:}5, 1{:}4, 3{:}2, 1{:}1\rangle) = 12$
{IT}	$\#\mathcal{W}(\langle 12; 1{:}5, 1{:}4, 3{:}2, 1{:}1\rangle) = 12$	$\#\mathcal{W}(\langle 12; 2{:}4, 1{:}3, 2{:}2, 1{:}1\rangle) = 11$

Only one Nash: $\alpha(FR) = \beta(FR) = 1/2$, $\alpha(IT) = \beta(IT) = 1/2$

$\#\mathcal{W}(\mathcal{U}) = 23/2$ and $\text{Act}(\mathcal{U}) = 23/2^7 \approx 0.1796$

Majority games with equal weights

Majority games with equal weights

Equal weight majority on n players game:

$$\Gamma(n,w) = \Gamma(w\lfloor n/2 \rfloor + 1, \underbrace{w, \ldots\ldots, w}_{n \text{ players}})$$

Coleman 1971:

$$\mathrm{Act}(\Gamma(n,w)) = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{2^n}\binom{n}{n/2}\right) & \text{for } n \text{ even} \\ 1/2 \ \text{ for } n \text{ odd} \end{cases}$$

A minimal egalitarian profile is

$$\mathcal{ME}(n, w, \delta, \mathcal{A}, \mathcal{D}) = \langle \Gamma(n,w), \mathcal{A}, \mathcal{D}, \delta_{\mathfrak{a}}, \delta_{\mathfrak{d}}, 1, 1 \rangle$$

where $\delta_{\mathfrak{a}}(i) = \delta$, for $i \in \mathcal{A}$, and $\delta_{\mathfrak{d}}(i) = -\delta$, for $i \in \mathcal{D}$.

Theorem 1

Let $n > 2$, $w > 1$, $0 < \delta < w$, and $\Gamma = \Gamma(n, w)$.

Let $\mathcal{A}, \mathcal{D} \subseteq [n]$ and $\mathcal{U} = \mathcal{ME}(n, w, \delta, \mathcal{A}, \mathcal{D})$.

Assume $\#\mathcal{A} > 0$, $\#\mathcal{D} > 0$. Then,

- if $\mathcal{A} = \mathcal{D}$, $\textsc{pne}(\Gamma(\mathcal{U})) = \emptyset$,
- if $\mathcal{A} \neq \mathcal{D}$ and $\mathcal{A} \subseteq \mathcal{D}$, $\textsc{pne}(G(\mathcal{U})) = \{(\{i\}, \{i\}) \mid i \in \mathcal{A}\}$,

$$\mathrm{Act}(\mathcal{U}) = \mathrm{Act}(\Gamma(n, w))$$

otherwise $\textsc{pne}(G(\mathcal{U})) = \{(\{i\}, \{j\}) \mid i \in \mathcal{A} \setminus \mathcal{D}, j \in \mathcal{D}\}$,

$$\mathrm{Act}(\mathcal{U}) = \mathrm{Act}(\Gamma(n, w)) + \frac{1}{2^n}\binom{n-2}{\lfloor n/2 \rfloor - 1}$$

Example 4 (Case $\mathcal{A} = \mathcal{D}$)

Let $n > 2$ and $w > 1$. Let $\Gamma = \Gamma(n, w)$ and

$$\mathcal{U} = \mathcal{ME}(n, w, 1, \{1, 2\}, \{1, 2\})$$

$G(\mathcal{U})$ has no PNE (Theorem 1). As $b_{\mathfrak{a}} = b_{\mathfrak{d}} = 1$ we have $A_{\mathfrak{a}} = A_{\mathfrak{d}} = \{\{1\}, \{2\}\}$ and $\mathfrak{a}$'s payoff matrix is:

	$\{1\}$	$\{2\}$
$\{1\}$	$\#\mathcal{W}(\Gamma(n, w))$	$\#\mathcal{W}(\Gamma(n, w) + \frac{1}{2^n}\binom{n-2}{\lfloor n/2 \rfloor - 1}$
$\{2\}$	$\#\mathcal{W}(\Gamma(n, w) + \frac{1}{2^n}\binom{n-2}{\lfloor n/2 \rfloor - 1}$	$\#\mathcal{W}(\Gamma(n, w))$

$$\mathrm{Act}(\mathcal{U}) = \mathrm{Act}(\Gamma(n, w)) + \frac{1}{2^{n-1}}\binom{n-2}{\lfloor n/2 \rfloor - 1}$$

Computational complexity considerations

Theorem 2

- Computing $\#\mathcal{W}(\Gamma)$, given Γ, is #P-complete.
- The following problems are NP-hard:
  - given Γ and Γ'; deciding whether $\mathrm{Act}(\Gamma) \neq \mathrm{Act}(\Gamma')$,
  - given $\mathcal{U}$ and $(a, d) \in A_{\mathfrak{a}} \times A_{\mathfrak{d}}$, deciding if d is a best response to a in $G(\mathcal{U})$;
  - given $\mathcal{U}$ associated to Γ, deciding whether $\mathrm{Act}(\mathcal{U}) \neq \mathrm{Act}(\Gamma)$.

A study based on the Council of the EU

Perturbations

Weights of the funding states in Γ_{EC6}, Γ_{EC27} and Γ_{EC12}

Γ	DE	FR	IT	NL	BE	LU
EC6	4	4	4	2	2	1
EC12	10	10	10	5	5	2
EC27	29	29	29	13	12	4

The differences provide perturbations δ_{12-27} and δ_{12-6},

δ	DE	FR	IT	NL	BE	LU
δ_{12-27}	$19 = 29 - 10$	19	19	$8 = 13 - 5$	7	2
δ_{12-6}	$4 - 10 = -6$	-6	-6	$2 - 3 = -3$	-3	-1

Fixed Quota

$\Gamma_{EC12} = \langle q, w_1, \ldots, w_{12} \rangle$, $q = 54$, $\Gamma_{EC12}[a, d] = \langle q, w'_1, \ldots, w'_{12} \rangle$.

$$\mathcal{U}_{12}(b_{\mathfrak{a}}, b_{\mathfrak{d}}) = \langle \Gamma_{EC12}, [6], [6], \delta_{12-27}, \delta_{12-6}, b_{\mathfrak{a}}, b_{\mathfrak{d}} \rangle$$

We provide $\text{Act}(\mathcal{U}_{12}(b_{\mathfrak{a}}, b_{\mathfrak{d}}))$ for each combination of $(b_{\mathfrak{a}}, b_{\mathfrak{d}})$,

FQ	0	1	2	3	4	5	6
0	0,098	0,049	0,016	0,002	0,000	0,000	0,000
1	0,405	0,339	0,262	0,177	0,145	0,115	0,105
2	0,573	0,513	0,449	0,385	0,348	0,319	0,311
3	0,666	0,636	0,603	0,566	0,537	0,512	0,509
4	0,722	0,697	0,668	0,637	0,608	0,578	0,575
5	0,762	0,737	0,711	0,682	0,658	0,637	0,631
6	0,773	0,749	0,723	0,695	0,679	0,654	0,645

An increase of power by $\mathfrak{a}$ results in an increase of Act.

Proportional Quota

Given $\Gamma_{EC12} = \langle q, w_1, \ldots, w_{12} \rangle$, $q = 54$ we consider:
$\Gamma^p_{EC12}[a, d] = \langle q', w'_1, \ldots, w'_n \rangle$ where $q' = \frac{q}{w([n])} w'([n])$.

PQ	0	1	2	3	4	5	6
0	0,098	0,098	0,095	0,088	0,092	0,101	0,104
1	0,151	0,156	0,158	0,158	0,163	0,172	0,183
2	0,182	0,175	0,164	0,155	0,160	0,167	0,172
3	0,170	0,163	0,165	0,156	0,161	0,166	0,172
4	0,155	0,152	0,151	0,141	0,143	0,150	0,156
5	0,141	0,142	0,136	0,134	0,131	0,133	0,141
6	0,134	0,131	0,125	0,124	0,118	0,127	0,131

In the proportional quota model it is not true that an increase of power by $\mathfrak{a}$ results in an increase of Act.

Same Spread & Reversed Roles

S= same spread, *D*= disjoint, *I*= Intersection, *r*=reversed

$$\begin{aligned} \mathcal{U}_{12SD}(b) &= \langle \Gamma_{\mathsf{EC}12}, \{0,1,2\}, \{3,4,5\}, \delta_{12-27}, \delta_{12-6}, b, b \rangle \\ \mathcal{U}_{12SI}(b) &= \langle \Gamma_{\mathsf{EC}12}, \{0,1,3\}, \{3,4,5\}, \delta_{12-27}, \delta_{12-6}, b, b \rangle \end{aligned}$$

In $\mathcal{U}_{12SDr}(b)$ and $\mathcal{U}_{12SIr}(b)$ we take $\delta_{\mathfrak{a}} = \delta_{12-6}$, $\delta_{\mathfrak{d}} = \delta_{12-27}$.

FQ	1	2	3
ec12SD	0,381	0,516	0,602
ec12SDr	0,390	0,528	0,602
ec12SI	0,381	0,516	0,602
ec12SIr	0,198	0,432	0,602

PQ	1	2	3
ec12SD	0,163	0,180	0,148
ec12SDr	0,164	0,188	0,148
ec12SI	0,163	0,180	0,173
ec12SIr	0,111	0,167	0,173

By reversing we get different Nash equilibria.

Minimal Egalitarian Profiles

δ	DE	FR	IT	NL	BE	LU
δ_1	1	1	1	1	1	1
δ_{-1}	-1	-1	-1	-1	-1	-1

Uncertainty model with unit perturbations

$$\mathcal{U}_x(b) = \langle \Gamma_{\text{ECx}}, \{0,1,2\}, \{3,4,5\}, \delta_1, \delta_{-1}, b, b \rangle$$

b	0	1	2	3
ec6	0,22	0,23	0,24	0,22
ec9	0,27	0,26	0,25	0,25
ec10	0,39	0,41	0,41	0,41
ec12	0,10	0,10	0,10	0,10
ec15	0,04	0,04	0,04	0,04
ec25-1	0,03	0,03	0,03	0,03
ec25-2	0,04	0,04	0,04	0,04
ec27	0,02	0,02	0,02	0,02

The total weights of the players is preserved, thus the fixed and proportional models are equivalent. Act does not present big variations.

Future extensions

- We would like to extend the framework to voting systems with uncertainty in the level of abstention.
- In Coleman 1971 two other measures were defined and merits to be considered:
  - The *power to initiate action*,

    $$\text{Initiate}_i(\Gamma) = \#\{S \in \mathcal{L}(\Gamma) \mid S \cup \{i\} \in \mathcal{W}(\Gamma)\}/\#\mathcal{L}(\Gamma)$$

    which gives the
  - The *power to prevent action*,

    $$\text{Prevent}_i(\Gamma) = \#\{S \in \mathcal{W}(\Gamma) \mid S \setminus \{i\} \in \mathcal{L}(\Gamma)\})/\#\mathcal{W}(\Gamma)$$