Probability Measures in Gödel$_\Delta$ Logic

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- Gödel Logics and Algebras;
  - Gödel$_\Delta$ Logic and Algebras;
  - Free Gödel$_\Delta$ Algebras;
- States over Free Gödel$_\Delta$ Algebras;
- Combinatorial Characterisation of States;
- Adaptation/Generalizations.

Gödel logic G can be semantically defined as a many-valued logic.
Let $\textsc{Form}$ be the set of formulas over propositional variables $x_1, x_2, , \ldots$ in the language $\vee, \wedge, \rightarrow, \neg, \bot$.

An assignment is a function $\mu : \textsc{Form} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \textsc{Form}$,

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\begin{aligned}
\mu(\alpha \wedge \beta) &= min\{\mu(\alpha), \mu(\beta)\}, \\
\mu(\alpha \vee \beta) &= max\{\mu(\alpha), \mu(\beta)\}, \\
\mu(\alpha \rightarrow \beta) &= \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases} \\
\mu(\neg\alpha) &= \mu(\alpha \rightarrow \bot), \\
\mu(\bot) &= 0, \\
\mu(\top) &= 1.
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$$\mu(\Delta(\alpha)) = \begin{cases} 1 & \text{if } \mu(\alpha) = 1 \\ 0 & \text{otherwise} \end{cases}$$

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A **tautology** is a formula α such that $\mu(\alpha) = 1$ for every assignment μ (denoted $\vDash \alpha$).

We write $\vdash \alpha$ to mean that α is derivable from the axioms of G_Δ using *modus ponens* as the only deduction rule.

G_Δ is complete with respect to the many-valued semantics defined above: in symbols, $\vdash \alpha$ if and only if $\vDash \alpha$.

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity equation:

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An **MTL algebra A** $= (A, \wedge, \vee, \odot, \to, \bot, \top)$ is a commutative integral bounded residuated lattice satisfying the **prelinearity** equation,

$$(x \to y) \vee (y \to x) = \top$$

.

A **Gödel Algebra A** $= (A, \wedge, \vee, \to, \bot, \top)$ is an **idempotent** MTL Algebra.

The variety $\mathbb{G}_\Delta$ is axiomatised as follows,

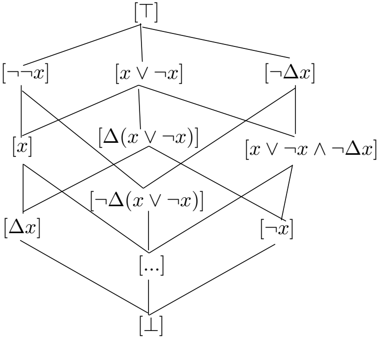
$$\Delta(x) \sqcup \neg\Delta(x) = 1, \qquad \Delta(x \sqcup y) \Rightarrow (\Delta(x) \sqcup \Delta(y)) = 1, \qquad \Delta(x) \Rightarrow x = 1,$$
$$\Delta(x) \Rightarrow \Delta(\Delta(x)) = 1, \qquad \Delta(x \Rightarrow y) \Rightarrow (\Delta(x) \Rightarrow \Delta(y)) = 1.$$

As usual, $\varphi, \psi \in \mathrm{FORM}_n$ are called **logically equivalent** $\varphi \equiv \psi$, if both $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$ hold.

The quotient set $\mathrm{FORM}_n/\equiv$ endowed with operations $\wedge, \vee, \to, \boldsymbol{\Delta}, \top, \bot$ induced from the corresponding logical connectives becomes a Gödel$_\Delta$ algebra with top and bottom element $\top$ and $\bot$, respectively.

The specific Gödel$_\Delta$ algebra $\mathcal{G}^n_\Delta = \mathrm{FORM}_n/\equiv$ is, by construction, the **Lindenbaum algebra** of G_Δ over the language $\{x_1, \ldots, x_n\}$.

The free 1-generated Gödel$_\Delta$ algebra $\mathcal{G}^1_\Delta$:



Lindenbaum algebras are isomorphic to free algebras, and then $\mathcal{G}^n_\Delta$ is the free n-generated Gödel$_\Delta$ algebra $\mathbf{F}^\Delta_n$.

Since the variety of Gödel$_\Delta$ algebras is locally finite, every finite Gödel$_\Delta$ algebra can be obtained as a quotient of a free n-generated Gödel$_\Delta$ algebra.

$\mathbf{F}_n^{\Delta}$ is isomorphic to the subalgebra of the algebra of all functions $f\colon [0,1]_{\Delta}^n \to [0,1]_{\Delta}$ generated by the projection $\overline{x_i}\colon (t_1,\ldots,t_n) \mapsto t_i$, for all $i \in \{1,2,\ldots,n\}$.
We write $\overline{\varphi}$ for the elements of $\mathbf{F}_n^{\Delta}$.

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The relation $\approx$ on $[0,1]^n$ is defined as:
$\mathbf{u} = (u_1,\cdots,u_n), \mathbf{v} = (v_1,\cdots,v_n) \in [0,1]^n$
$\mathbf{u} \approx \mathbf{v}$ iff there is a permutation σ of $\{1,\ldots,n\}$
and a map $\prec\colon \{0,\ldots,n\} \to \{<,=\}$ such that

$$0 \prec_0 u_{\sigma(1)} \prec_1 \cdots \prec_{n-1} u_{\sigma(n)} \prec_n 1$$

iff

$$0 \prec_0 v_{\sigma(1)} \prec_1 \cdots \prec_{n-1} v_{\sigma(n)} \prec_n 1$$

$\approx$ is an equivalence relation and $[\mathbf{u}]$ is the equivalence class of $\mathbf{u}$.
$[0,1]^n/\approx$ is hence a partition of $[0,1]^n$.

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With each class $[\mathbf{u}]$, where
$0 \prec_0 u_{\sigma(1)} \prec_1 \cdots \prec_{n-1} u_{\sigma(n)} \prec_n 1$,
we associate a unique **ordered partition** $\rho_{\mathbf{u}} = Q_1 < \cdots < Q_h$ of the set $\{\bot, x_1,\dots,x_n,\top\}$ in the following way:

- $\bot \in Q_1$; $\top \in Q_h$; $h > 1$;
- if $\prec_i$ is $=$ then $x_{\sigma(i)}, x_{\sigma(i+1)} \in Q_j$;
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Ordered partitions are in bijections with equivalence classes $[\mathbf{u}] \in [0,1]^n/\approx$.
When $\rho = \rho_{\mathbf{u}}$, we denote by D_ρ the associated equivalence class $[\mathbf{u}]$.
We write Ω_n for the set of all ordered partitions.

$\mathbf{F}_n^{\Delta}$ is isomorphic to the subalgebra of the algebra of all functions $f\colon [0,1]_{\Delta}^n \to [0,1]_{\Delta}$ generated by the projection $\overline{x_i}\colon (t_1,\ldots,t_n) \mapsto t_i$, for all $i \in \{1,2,\ldots,n\}$.
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A ***n*-variate G_{Δ}-function** is a function $f : [0,1]^n \to [0,1]$ such that for every $\mathbf{u} \in [0,1]^n$ (equivalently, for any $\rho \in \Omega_n$) the restriction of f to $[\mathbf{u}]$ (equivalently, to D_{ρ}) is either equal to 0, or to 1, or to a projection function $\overline{x_i}$.

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Theorem

The elements of $\mathbf{F}_n^{\Delta}$ are exactly the n-variate G_{Δ}-functions.

Let $x \lhd y = \Delta(x \to y) \wedge \neg\Delta(y \to x)$.
Interpreted in $[0, 1]$ we have
$\overline{x \lhd y} = 1$ if $x < y$ and $\overline{x \lhd y} = 0$ otherwise.

For any $\rho = \rho_{\mathbf{u}} \in \Omega_n$, consider the formula

$$\chi_\rho = \bigwedge_{i=0}^{n} \delta_i,$$

$$\delta_i = \begin{cases} \Delta(x_{\sigma(i)} \leftrightarrow x_{\sigma(i+1)}) & \text{iff } \prec_i \text{ is } =, \\ x_{\sigma(i)} \lhd x_{\sigma(i+1)} & \text{iff } \prec_i \text{ is } < . \end{cases}$$

Then it is straightforward to check that
$\overline{\chi_\rho}(\mathbf{v}) = 1$ iff $\mathbf{v} \approx \mathbf{u}$, while $\overline{\chi_\rho}(\mathbf{v}) = 0$ otherwise.

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For $n = 2$, the set of Gödel partitions Ω_2 is:

$$\begin{aligned}
\rho_1 &= \{0, x, y\} < \{1\} \\
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$$\overline{\chi_{\rho_1}}(\rho_1) = 1$$
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because $\Delta(x \leftrightarrow y) = \Delta(x) = 0$ on ρ_5.

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y_ρ is the element of $\{\bot, x_1, \dots, x_n, \top\}$ such that $\overline{y_\rho}$ coincides with f over the whole of D_ρ.

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$$\varphi = \bigvee_{\rho \in \Omega_n} (\chi_\rho \wedge y_\rho).$$

For any point $\mathbf{u} \in [0,1]^n$, $\overline{\varphi}(\mathbf{u})$ coincides with $\overline{\chi_\rho \wedge y_\rho}(\mathbf{u})$ for the unique $\rho \in \Omega_n$ such that $\mathbf{u} \in D_\rho$.
Then, $\overline{\varphi} = f$.

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A **state** on $\mathbf{F}_n^\Delta$ is a function $s\colon \mathbf{F}_n^\Delta \to [0,1]$ such that, for every $f, g \in \mathbf{F}_n^\Delta$:

1. $s(\bot) = 0$, $s(\top) = 1$;
2. $s(f \vee g) = s(f) + s(g) - s(f \wedge g)$;
3. If $f \leq g$ then $s(f) \leq s(g)$;
4. If $f \leq g$ and $s(g) = s(f)$ then $s(\Delta(g \to f)) = 1$.

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Theorem

The following hold.

1. *If $s\colon \mathbf{F}_n^\Delta \to [0,1]^n$ is a state, there exists a Borel probability measure μ on $[0,1]^n$ such that*
$$\int_{[0,1]^n} f \, \mathrm{d}\mu = s(f) \ , \quad \textit{for every } f \in \mathbf{F}_n^\Delta \ .$$
2. *Viceversa, for any Borel probability measure μ on $[0,1]^n$, the function $s\colon \mathbf{F}_n^\Delta \to [0,1]$ defined by the above integral is a state.*

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States of $\mathbf{F}_n^\Delta$ are the convex combinations of finitely many truth value assignments.

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Corollary

States of $\mathbf{F}_n^\Delta$ are the convex combinations of finitely many truth value assignments.

Let s be the state on $\mathbf{F}_2(\mathbb{G}_\Delta)$ given by

$s(\overline{\chi_{\rho_1}}) = 1/3$ $\quad s(\overline{\chi_{\rho_4}}) = 1/6$

$s(\overline{\chi_{\rho_5}}) = 1/2$ $\quad s(\overline{x \wedge \chi_{\rho_4}}) = 1/12$

$s(\overline{x \wedge \chi_{\rho_5}}) = 1/12$ $\quad s(\overline{y \wedge \chi_{\rho_4}}) = 1/6$

$s(\overline{\chi_\sigma}) = 0$ $\quad$ for $\sigma \notin \{\rho_1, \rho_4, \rho_5\}$

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Define the discrete measure μ by setting

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For $n = 2$, the set of Gödel partitions Ω_2 is:

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States of $\mathbf{F}_n^\Delta$ are the convex combinations of finitely many truth value assignments.

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$$\int_{[0,1]^2} f \, \mathrm{d}\mu = \sum_{i \in \{1,4,5\}} f(z_{\rho_i}) \mu(\{z_{\rho_i}\}) = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2} = s(f)\,.$$

A nonempty subset F of A is called an *upper-set* when for all $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$. If $x \odot y \in F$ for all $x, y \in F$, then F is a **filter** of **A**. We call $\bigwedge_{x \in F} x$ the *generator* of the filter F. A filter F of A is **prime** if $F \neq A$ and for all $x, y \in A$, either $x \to y \in F$ or $y \to x \in F$.

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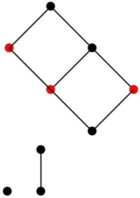
Let **A** be a finite Gödel algebra, then Spec**A** is a forest.



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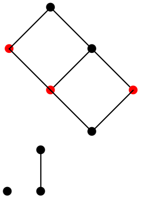

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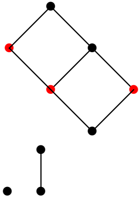

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Proposition (Aguzzoli and Codara, 2016)

Every finite G_Δ-algebra $\mathbf{A}$ is a direct product of chains. That is, $\mathbf{A} \simeq \prod_{F \in \mathrm{Max}(\mathbf{A})} \mathbf{A}/F$, and Max($\mathbf{A}$) = Spec($\mathbf{A}$).

For each $\mathbf{A} \in (\mathbb{G}_\Delta)_{fin}$, the poset $Spec(\bar{\mathbf{A}})$, that is, the prime spectrum of the G-algebra reduct of $\mathbf{A}$, ordered by reverse inclusion, is isomorphic with the poset of the j.i. elements of $\mathbf{A}$.

$$\mathrm{Spec}^{\Delta}(\mathbf{A}) = \mathcal{C}(Spec(\bar{\mathbf{A}}))$$

where $\mathcal{C}(P)$ is the multiset $\{C_1, C_2, \ldots, C_u\}$, when the poset P is a disjoint union $C_1 \cup C_2 \cup \cdots \cup C_u$ of chains.

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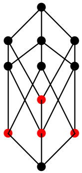

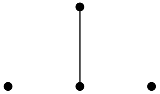

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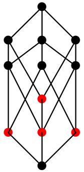

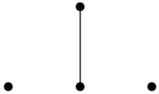

Conversely, given a chain C we define:

$\mathrm{Sub}^\Delta(C) =$
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where $\Delta C = C$ and $\Delta D = \emptyset$, for each subchain $D \subsetneq C$,
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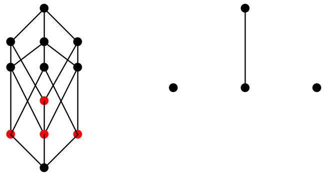

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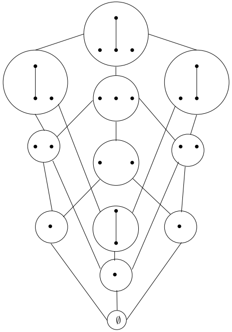

MC be the category whose objects are finite multisets of (nonempty) finite chains, and whose morphisms $h\colon C \to D$, are defined as follows.
Display C as $\{C_1, \ldots, C_m\}$ and D as $\{D_1, \ldots, D_n\}$. Then $h = \{h_i\}_{i=1}^m$, where each h_i is an order preserving surjection $h_i\colon C_i \twoheadrightarrow D_j$ for some $j \in \{1, 2, \ldots, n\}$.

$(\mathbb{G}_\Delta)_{fin}$ is the full subcategory of $\mathbb{G}_\Delta$ whose objects have finite cardinality, and morphisms are simply homomorphisms of algebras.

The two previous constructions are functorial:

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Theorem (Aguzzoli and Codara, 2016)

The categories $(\mathbb{G}_\Delta)_{fin}$ and MC *are dually equivalent.*

A **labeling** l is a function $l\colon \mathrm{Spec}^{\Delta}\,\mathbf{F}_n^{\Delta} \to [0,1]$, such that

1. $\sum_{\mathfrak{p}\in \mathrm{Spec}^{\Delta}\,\mathbf{F}_n^{\Delta}} l(\mathfrak{p}) = 1$;
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Theorem

Let S_n be the collection of all states $s\colon \mathbf{F}_n^{\Delta} \to [0,1]$, and let L_n be the collection of all labelings $l\colon \mathrm{Spec}^{\Delta}\,\mathbf{F}_n^{\Delta} \to [0,1]$. Then, the map defined for every formula φ over the set of variables $\{x_1,\ldots,x_n\}$ by

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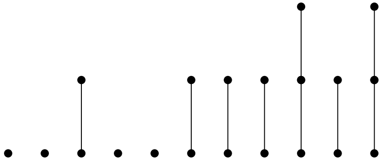

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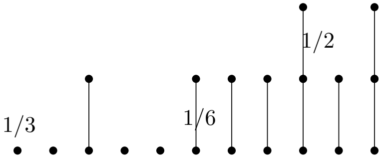

Drastic Product algebras constitute the subvariety $\mathbb{DP}$ of $\mathbb{MTL}$ axiomatised by $x \sqcup \sim(x * x) = 1$.

Theorem (Aguzzoli, Bianchi and V., 2014)

$\mathsf{MC}^{\top}$ *is dually equivalent to the category* $\mathbb{DP}_{fin}$ *of finite DP algebras and their homomorphisms.*

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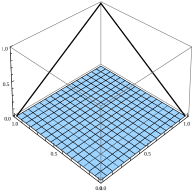

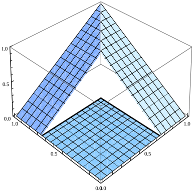

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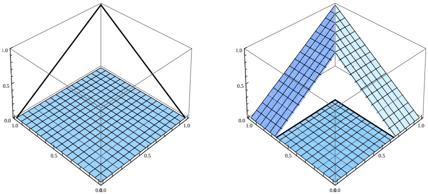

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