Solving trajectory optimization problems by influence diagrams

Jirka Vomlel and Václav Kratochvíl,
Czech Academy of Sciences, Prague

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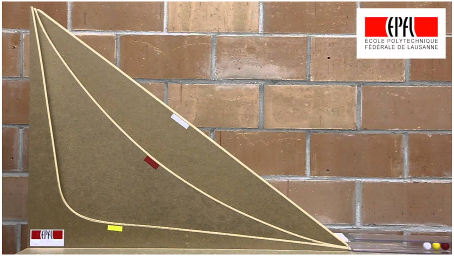

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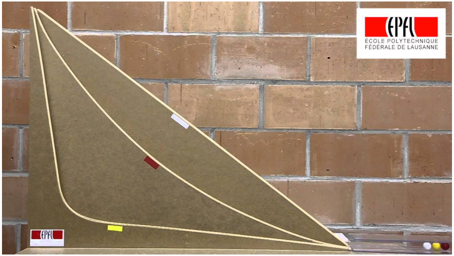

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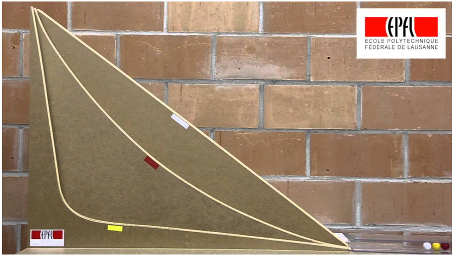

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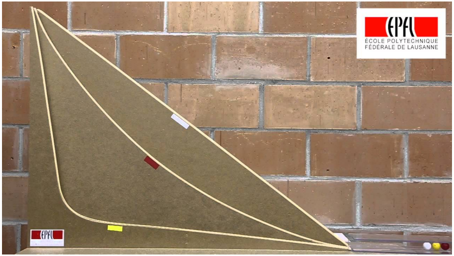

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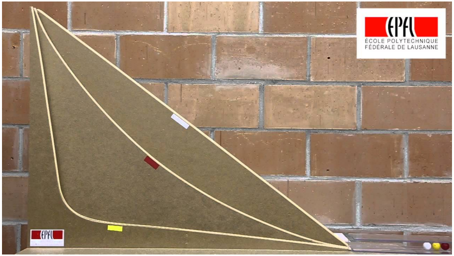

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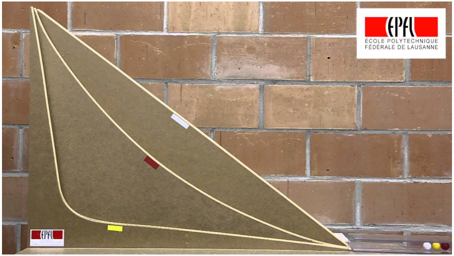

See the video of the experiment.

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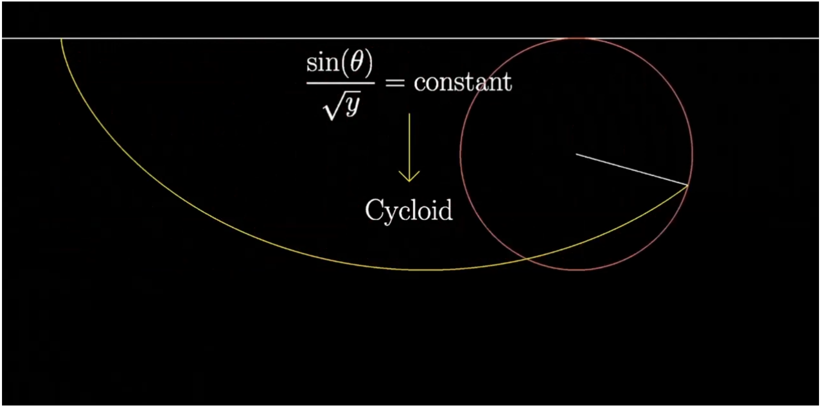
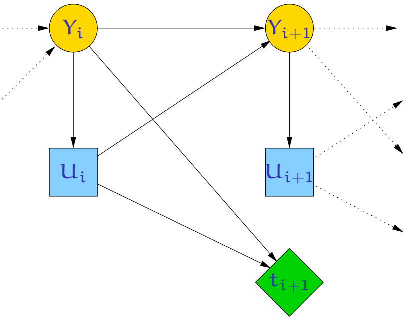
Cycloid - the optimal solution



Influence diagram for the Brachistochrone problem



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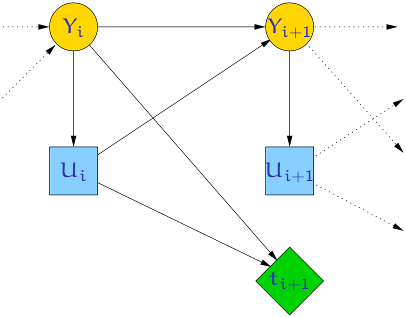

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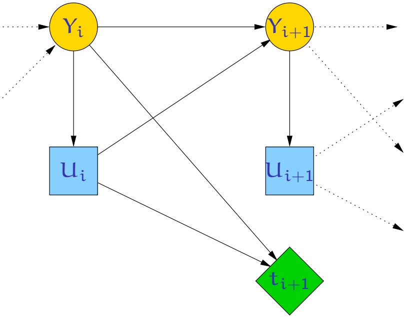

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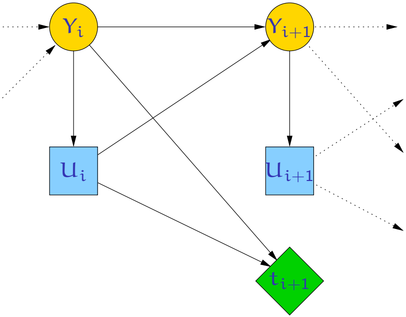

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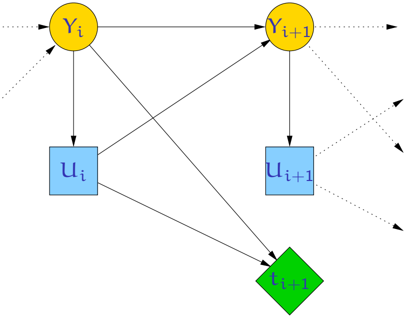

- Discretize the x-coordinate to segments of length Δx.
- State variable Y_i – the vertical position of the mass point.
- Decision variable U_i – the vertical shift.
- Utility node $t_{i+1}(y_i, u_i)$ – the utility function is the time :

$$t_{i+1} = \begin{cases} \dfrac{\Delta x}{\sqrt{-2 \cdot g \cdot y_i}} & \text{if } u_i = 0 \\ -\sqrt{\dfrac{2}{g}} \cdot \dfrac{\sqrt{(\Delta x)^2 + u_i^2}}{u_i} \cdot \left(\begin{array}{c} \sqrt{-y_i} \\ -\sqrt{-u_i - y_i} \end{array} \right) & \text{otherwise.} \end{cases}$$

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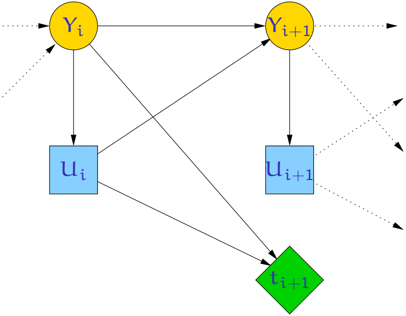

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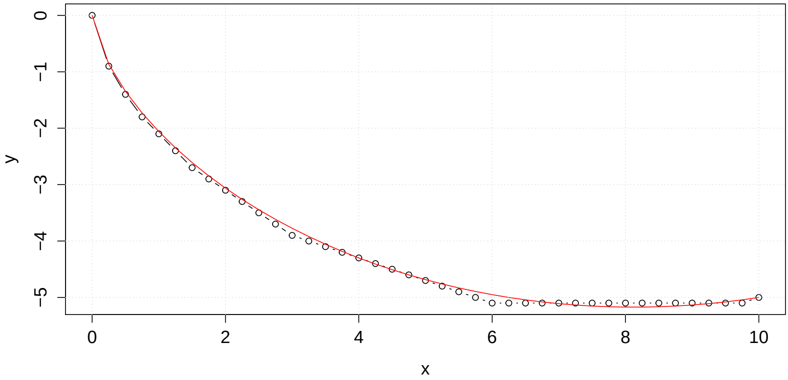
The goal is to minimize total time $\sum_{i=0}^{n} t_{i+1}$.

Comparison of the optimal solution with the influence diagram solution



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- the engine thrust is bounded.

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$$\frac{dv}{dh} = f(h, m, v)$$

"equlibrium of forces"

$$= -\frac{1}{m} \cdot \left(\frac{c \cdot u}{v} + \frac{1}{2} \cdot s \cdot c_D \cdot \rho_0 \cdot \exp\left(\beta \cdot (1 - h)\right) \cdot v \right) - \frac{1}{v} \cdot \frac{1}{h^2}$$

Optimal solution

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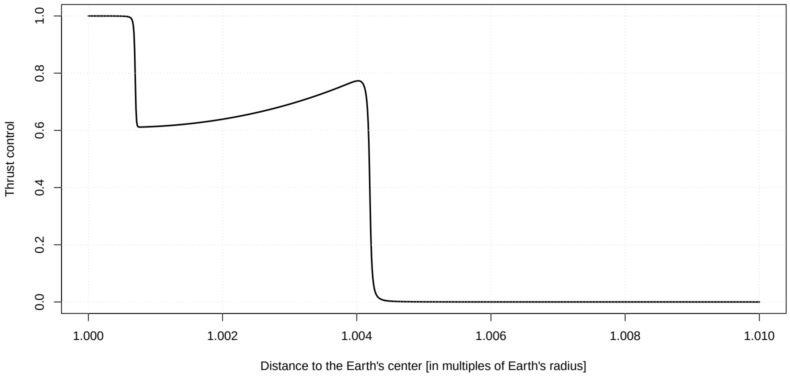
(c) coasting subarcs (i.e. subarcs with the zero thrust).

Optimal solution

(found by Bocop, using a NLP solver IPOPT)



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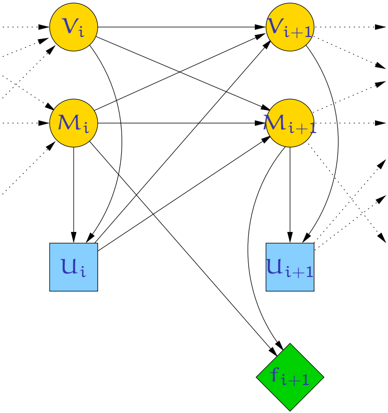

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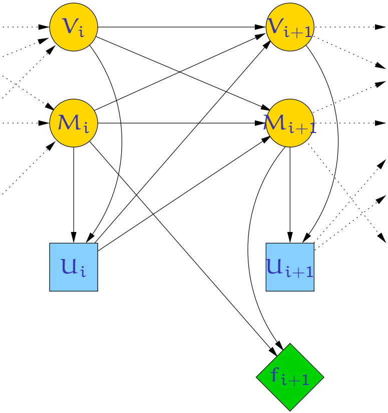

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  M_i – the rocket mass (payload + fuel),
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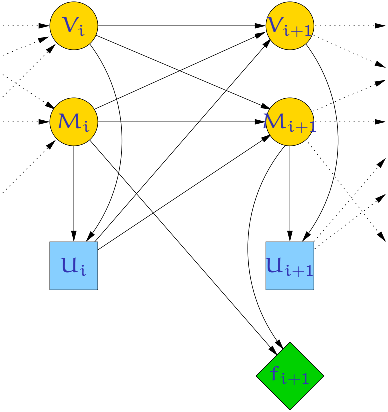

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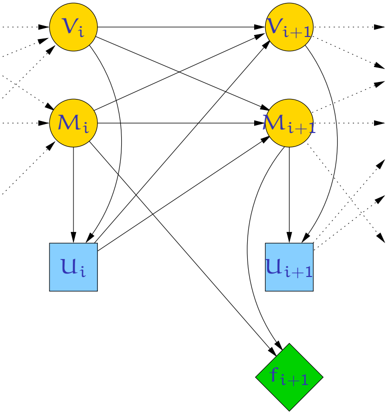

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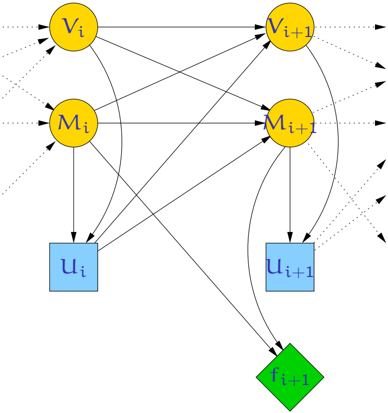

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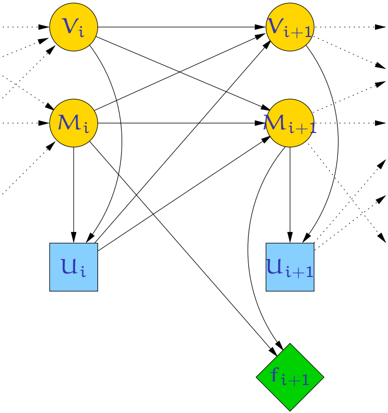

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- In CPTs a stochastic approximation of the state transitions by a probability mixture of two nearest states is used.

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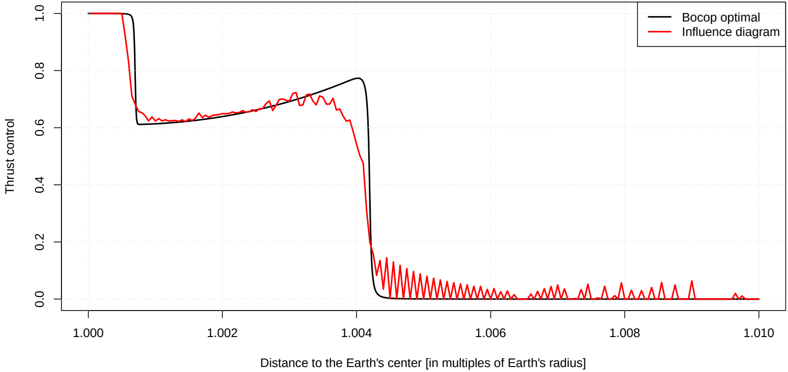

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6. Construct the ID having in each segment:
   - a chance node for each state variable,
   - a decision node for each control variable, and
   - a utility node.

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10. Find and store the optimal policy for each segment of the trajectory by solving the ID.
11. During the application the optimal policy for the actual observed values of state variables at each point of the trajectory is used.
12. If the controlled object deviates from the optimal solution use the stored optimal policy for the observed state.

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- Thanks to the decomposability of the optimization criteria computations are performed locally in the cliques.
- In influence diagrams it is easy to modify the optimality criteria – as long as it decomposes additively along the path.
- Future work: influence diagrams with continuous variables.